

High dimensional deformed rectangular matrices with applications in matrix denoising

Xiukai Ding *

Department of Statistical Sciences, University of Toronto

Abstract

We consider the recovery of a low rank $M \times N$ matrix S from its noisy observation \tilde{S} in two different regimes. Under the assumption that M is comparable to N , we propose two consistent estimators for S . Our analysis relies on the local behavior of the large dimensional rectangular matrices with finite rank perturbation. We also derive the convergent limits and rates for the singular values and vectors of such matrices.

1 Introduction

Matrix denoising is important in many scientific endeavors. They appear prominently in signal processing [25], image denoising [10], machine learning [26], statistics [11, 12, 14], empirical finance [17] and biology [21]. In these applications, researchers are interested in recovering the true deterministic matrix from a noisy observation. Consider that we can observe a noisy $M \times N$ data matrix \tilde{S} , where

$$\tilde{S} = X + S. \quad (1.1)$$

In model (1.1), the deterministic matrix S is known as the signal matrix and X the noise matrix. In the classic framework, under the assumption that M is much smaller than N , the truncated singular value decomposition (TSVD) is the default technique, see for example [1, 13]. This method recovers S with an estimator \hat{S} using the truncated singular value decomposition of \tilde{S} : write

$$\hat{S} = \sum_{i=1}^m \mu_i \tilde{u}_i \tilde{v}_i^*,$$

where $m < \min\{M, N\}$ denotes the truncated level, $\mu_i, \tilde{u}_i, \tilde{v}_i$, $i = 1, 2, \dots, m$ are the singular values, left singular vectors and right singular vectors of \tilde{S} respectively. We usually need to provide a thresholding γ to choose m and use the singular values only when $\mu_i \geq \gamma$. Two popular methods are the soft thresholding [9] and hard thresholding [11].

In recent years, the advance of technology has lead to the observation of massive scale data, where the dimension of the variable is comparable to the length of the observation. For example, the gene expression data [21] contains a large number of DNA sequences, which may be comparable to or even larger than the number of observations. In this situation, the TSVD will lose its validity. To address this problem, in the present paper, we consider the matrix denoising problem (1.1) by assuming M is comparable to N and estimate S in the following two regimes:

Regime (1). S is of low rank and we have prior information that its singular vectors are sparse;

*E-mail: xiukai.ding@mail.utoronto.ca.

Regime (2). S is of low rank and we have no prior information on the singular vectors.

In regime (1), S is called simultaneously low rank and sparse matrix. This type of matrix S has been heavily used in statistics, machine learning and biology. A typical example is from the study of gene expression data [21]. An microarray experiment typically assesses a large number of DNA sequences (genes, cDNA clones, or expressed sequence tags) under multiple conditions. The gene expression data from an microarray experiment can be represented by a real-valued expression matrix S , where the rows of S correspond to the expression pattern of genes (e.g. cancer patient) and column correspond to the gene levels. A subset of gene patterns can be clustered together as a subtype of the same pattern, which in turn is determined by a subset of genes. The original gene expression matrix obtained from a scanning process contains noise, missing values and systematic variations arising from the experimental procedure. Therefore, our discussion here provides an ideal model for the gene expression data. In [26], Yang, Ma and Buja also consider such problem but from a quite different perspective. They do not take the local behavior of singular values and vectors into consideration. Instead, they use an adaptive thresholding method to recover S in (1.1).

In regime (2), it is almost hopeless to completely recover S as we have little information of S . We are interested in looking at what is the best we can do in this case. A natural (and probably necessary) assumption is rotation invariance [6], as the only information we know about the singular vectors is orthonormality. We will propose a consistent rotation invariant estimator in this regime. It is notable that, in this case, our result coincides with the results proposed by Gavish and Donoho [12], where they consider the estimator from another perspective and restrict the estimator to be conservative (see Definition 3 in [12]).

Our methodologies rely on investigating the local properties of singular values and vectors. We systematically investigate the convergent limits and rates for the singular values and vectors for high dimensional rectangular matrices assuming M is comparable to N . The convergent limits are firstly computed by Benaych-Georges and Nadakuditi in [3] for (1.1) under the assumption that the distribution of the entries of X is bi-unitarily invariant (see Remark 2.6 in [3]). We generalize this result to more general distributions and further compute the convergent rates.

In this paper, we consider the problem (1.1) and assume that $X = (x_{ij})$ is an $M \times N$ matrix with i.i.d centered entries $x_{ij} = N^{-1/2}q_{ij}$, where q_{ij} is of unit variance and there exists a constant C , for some $p \in \mathbb{N}$ large enough, q_{ij} satisfies the following condition

$$\mathbb{E}|q_{ij}|^p \leq C. \quad (1.2)$$

We denote the SVD of S as $S = UDV^*$, where

$$D = \text{diag}\{d_1, \dots, d_r\}, \quad U = (u_1, \dots, u_r), \quad V = (v_1, \dots, v_r),$$

and where $u_i \in \mathbb{R}^M$, $v_i \in \mathbb{R}^N$ are orthonormal vectors and r is a fixed constant. We also assume $d_1 > d_2 > \dots > d_r > 0$. Then (1.1) can be written as

$$\tilde{S} = X + UDV^*. \quad (1.3)$$

Throughout the paper, we are interested in the following setup

$$c_N := \frac{N}{M}, \quad \lim_{N \rightarrow \infty} c_N = c \in (0, \infty). \quad (1.4)$$

It is well-known that for the noise matrix X , the spectrum of XX^* satisfies the celebrated Marchenko-Pastur (MP) law [19] and the largest eigenvalue satisfies the Tracy-Widom (TW) distribution [24]. Specifically, denote $\lambda_i := \lambda_i(XX^*)$, $i = 1, 2, \dots, K$, where $K = \min\{M, N\}$ as the eigenvalues of XX^* , we have that

$$\lambda_1 = \lambda_+ + O(N^{-2/3}), \quad \lambda_+ = (1 + c^{-1/2})^2, \quad (1.5)$$

holds with high probability. Furthermore, denote ξ_i, ζ_i as the singular vectors of X , then we have [7]

$$\max_i \{|\xi_i|^2 + |\zeta_i|^2\} = O(N^{-1}),$$

holds with high probability.

To sketch the behavior of \tilde{S} , we consider the case when $r = 1$ in (1.3). Assuming that the distribution of the entries of X is bi-unitarily invariant, Benaych-Georges and Nadakuditi establish the convergent limits in [3] using free probability theory. Denote $\mu_i := \mu_i(\tilde{S}\tilde{S}^*)$, $i = 1, 2, \dots, K$, as the eigenvalues of $\tilde{S}\tilde{S}^*$, they proved that when $d > c^{-1/4}$, μ_1 will detach from the spectrum of the MP law and become an outlier. And when $d < c^{-1/4}$, μ_1 converges to λ_+ and sticks to the spectrum of MP law. For the singular vectors, denote \tilde{u}_i , \tilde{v}_i as the left and right singular vectors of \tilde{S} , $i = 1, 2, \dots, K$. They prove that when $d > c^{-1/4}$, \tilde{u}_1 , \tilde{v}_1 will be concentrated on cones with axis parallel to u_1 , v_1 respectively, and the apertures of the cones converge to some deterministic limits. And when $d < c^{-1/4}$, \tilde{u}_1 , \tilde{v}_1 will be asymptotically perpendicular to u_1 , v_1 respectively.

We point out that these results have also been proved for the covariance matrices with multiplication perturbation. In the seminal paper [2], Baik, Ben-Arous and P      proved that when $\mathbb{E}X^*X \neq I$, some eigenvalues of XX^* will detach from the bulk and become outliers. This is the so-called BBP transition. For a comprehensive study about such models, we refer to [5, 20], where they systematically study the local behavior of eigenvalues and eigenvectors of covariance matrices. Our analysis of the singular values and vectors are based on their discussions.

Our computation and proofs rely on the celebrated isotropic local MP law [4, 16] and the anisotropic law [15]. These results say that the eigenvalue distribution of the sample covariance matrix XX^* is close to the MP law, down to the spectral scales containing slightly more than one eigenvalue. These local laws are formulated using the Green functions,

$$\mathcal{G}_1(z) := (XX^* - z)^{-1}, \quad \mathcal{G}_2(z) := (X^*X - z)^{-1}, \quad z = E + i\eta \in \mathbb{C}^+. \quad (1.6)$$

The above local MP laws have many applications in the local analysis of sample covariance matrices. To list a few, the rigidity of the eigenvalues [15], the completely delocalization of singular vectors [7], the edge and bulk universality of sample covariance matrices [4, 8, 15, 16].

To illustrate our results and ideas, we give an overview and a heuristic description of the local behavior of singular values and vectors of \tilde{S} and how they can be used to recover the signal matrix S in (1.1). Our first step is to construct a Hermitian matrix. As we have seen from [7, 8], the self-adjoint linearization technique is quite useful in dealing with rectangular matrices. Hence, in a first step, we denote by

$$\tilde{H} = \begin{bmatrix} 0 & z^{1/2}\tilde{S} \\ z^{1/2}\tilde{S}^* & 0 \end{bmatrix} = \begin{bmatrix} 0 & z^{1/2}X \\ z^{1/2}X^* & 0 \end{bmatrix} + \begin{bmatrix} 0 & z^{1/2}UDV^* \\ z^{1/2}VDU^* & 0 \end{bmatrix} = H + \mathbf{U}\mathbf{D}\mathbf{U}^*, \quad (1.7)$$

where \mathbf{D}, \mathbf{U} are defined by

$$\mathbf{D} := \begin{bmatrix} 0 & z^{1/2}D \\ z^{1/2}D & 0 \end{bmatrix}, \quad \mathbf{U} := \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}, \quad (1.8)$$

and $z \in \mathbb{C}^+$. (1.7) is a very convenient expression. On one hand, the eigenvalues of $\tilde{S}\tilde{S}^*$ can be uniquely characterized by the eigenvalues of \tilde{H} (see the discussion after [8, (2.21)]). On the other hand, the Green functions of XX^* and X^*X are contained in that of H (see (4.14)). Thus a control of the Green functions of H will yield a control of those of XX^* and X^*X . Throughout this paper, we will use

$$\lambda_1 \geq \dots \geq \lambda_K, \quad K = \min\{M, N\}, \quad (1.9)$$

to represent the eigenvalues of XX^* and denote

$$\mu_1 \geq \dots \geq \mu_K,$$

as the eigenvalues of $\tilde{S}\tilde{S}^*$. We also denote ξ_i , ζ_i , $i = 1, \dots, K$ as the singular vectors of X and \tilde{u}_i , \tilde{v}_i as the singular vectors of \tilde{S} . And we denote $G(z)$ as the Green functions of H , $\tilde{G}(z)$ as that of \tilde{H} .

Consider $r = 1$ in (1.3), by a simple perturbation discussion (see Lemma 4.11), we find that μ_1 satisfies the following deterministic equation

$$\det(\mathbf{U}^*G(\mu_1)\mathbf{U} + \mathbf{D}^{-1}) = 0. \quad (1.10)$$

Using a slightly modified anisotropic law in [15], we find that (see Lemma 4.13), G has a deterministic limit Π when N is large enough. Heuristically, this implies that (1.10) still holds true when we replace G with Π

when N is large enough. An elementary calculation shows that, when $d > c^{-1/4}$, $\mu_1 \rightarrow p(d)$, where $p(d)$ is defined as

$$p(d) = \frac{(d^2 + 1)(d^2 + c^{-1})}{d^2}. \quad (1.11)$$

When $d > c^{-1/4}$, the largest eigenvalue μ_1 will detach from the bulk and become an outlier around its *classical location* $p(d)$. We would expect this happens under a scale of $N^{-1/3}$. This can be understood in the following ways: increasing d beyond the critical value $c^{-1/4}$, we expect μ_1 to become an outlier, where its location $p(d)$ is located at a distance greater than $O(N^{-2/3})$ from λ_+ . By using mean value theorem, the phase transition will take place on the scale when

$$|d - c^{-1/4}| \geq O(N^{-1/3}). \quad (1.12)$$

We prove that when (1.12) happens, with high probability

$$\mu_1 = p(d) + O\left(N^{-1/2}(d - c^{-1/4})^{1/2}\right). \quad (1.13)$$

Below this scale, we would expect the spectrum of $\tilde{S}\tilde{S}^*$ will be sticking to that of XX^* . Especially, the largest eigenvalue μ_1 still has the Tracy-Widom distribution with the scale $N^{-2/3}$, which reads as

$$\mu_1 = \lambda_+ + O(N^{-2/3}). \quad (1.14)$$

For the singular vectors, when $d > c^{-1/4}$, we have that

$$\langle u_1, \tilde{u}_1 \rangle^2 \rightarrow a_1(d), \quad \langle v_1, \tilde{v}_1 \rangle^2 \rightarrow a_2(d),$$

where $a_1(d)$, $a_2(d)$ are deterministic functions of d and defined in (2.12). For the local behavior, we will use an integral representation of Greens functions (see (6.32)). However, when $r > 1$, if $d_i \approx d_j$, $i \neq j$, we would expect that $\tilde{u}_i(\tilde{v}_i)$, $\tilde{u}_j(\tilde{v}_j)$ lie in the same eigenspace. And then we can not distinguish the singular vectors. Therefore, in this paper, we assume that for $i \neq j$, there exists some $\epsilon_0 > 0$, such that d_i, d_j satisfy the following condition

$$|p(d_i) - p(d_j)| \geq N^{-1/2+\epsilon_0}(d_i - c^{-1/4})^{1/2}. \quad (1.15)$$

(1.15) is referred as non-overlapping condition in [5, 16], it ensures that the eigenspace corresponding to different d_i , $i = 1, \dots, r$ can be well separated. This can be understood in the following ways: when $d_i, d_j > c^{-1/4}$, the corresponding eigenvalues μ_i, μ_j of $\tilde{S}\tilde{S}^*$ will converge to $p(d_i)$ and $p(d_j)$ respectively. Hence, (1.15) ensures that the singular vectors can be distinguished individually. (1.15) can be written as

$$|d_i - d_j| \geq N^{-1/2+\epsilon_0}(d_i - c^{-1/4})^{-1/2}, \quad (1.16)$$

by using mean value theorem. Under the assumption that d_i 's are well-separated and above the scale (1.12), we prove that

$$\langle u_1, \tilde{u}_1 \rangle^2 = a_1(d) + O(N^{-1/2}), \quad \langle v_1, \tilde{v}_1 \rangle^2 = a_2(d) + O(N^{-1/2}). \quad (1.17)$$

Below the scale of (1.12), we prove that

$$\langle u_1, \tilde{u}_1 \rangle^2 = O(N^{-1}), \quad \langle v_1, \tilde{v}_1 \rangle^2 = O(N^{-1}). \quad (1.18)$$

In the present paper, for the discussion of singular vectors, we use (1.16) as an assumption just for the purpose of statistical estimation of (1.1). It has been proved in [5, Section 5.2], the non-overlapping condition can be removed with extra work. We will not pursue this generalization.

Armed with (1.13), (1.14), (1.17) and (1.18), we can go to the matrix denoising problem (1.1) under the two different regimes. In the first regime, we assume there exists sparsity structure of the singular vectors, in the case when $d > c^{-1/4}$, we would expect \tilde{u}_1, \tilde{v}_1 to be sparse as well. Hence, \tilde{S} will be of sparse structure. Therefore, by suitably choosing a submatrix of \tilde{S} and doing SVD for the submatrix, we can get an estimator for the singular vectors. Compared to the machine learning approach [12, 26], our novelty is to truncate the singular values and vectors simultaneously. For the estimation of singular values, we can reverse (1.13) to get the estimator for d . For the singular vectors, based on (1.18), the threshold should be much larger

than $N^{-1/2}$ and we will use the K-means clustering algorithm to choose such thresholds. However, when $d < c^{-1/4}$, we can estimate nothing according to (1.14) and (1.18).

In the second regime, as we have no prior information whatsoever on the true eigenbasis of S , the only possibility is to use the eigenbasis of \tilde{S} . This is equivalent to the assumption of rotation invariance. We will propose a consistent rotation invariant estimator (RIE) $\Xi(\tilde{S})$, which satisfies the following condition,

$$\Omega_1 \Xi(\tilde{S}) \Omega_2 = \Xi(\Omega_1 \tilde{S} \Omega_2), \quad (1.19)$$

where Ω_1, Ω_2 are rotation matrix in O_M, O_N respectively. We will provide such an estimator in Section 2.2.

Before concluding this section, we outline our main contributions of this paper:

(i). We systematically study the local behavior of singular values and vectors for finite rank perturbation of large dimensional rectangular matrices of model (1.1). We compute the convergent limits and rates for them. When (1.12) and (1.15) hold true, the singular values and vectors detach from the bulk and become outliers, the results are recorded by (1.13) and (1.17). Below the scale of (1.12), the singular values and vectors will be stucked to that of the MP law and the behavior are read as (1.14) and (1.18).

(ii). We propose two consistent estimators for the matrix denoising model (1.3) under two different regimes. We provide practical algorithms to compute the optimal estimators. For the sparse estimation, as far as we know, our paper is the first one to truncate the singular values and vectors simultaneously.

This paper is organized as follows. In Section 2, we propose the estimators for (1.3) under two regimes. Numerical algorithms and simulations are provided to support our discussions. In Section 3, we give the main results of this paper. In Section 4, we record the basic tools for the proofs of the main theorems. In Section 5 and 6, we prove the main theorems listed in Section 3.

Conventions. All quantities that are not explicitly constants may depend on N , and we usually omit N from our notations. We use C to denote a generic large positive constant, whose value may change from one line to the next. Similarly, we use ϵ to denote a generic small positive constant. For two quantities a_N and b_N depending on N , the notation $a_N = O(b_N)$ means that $|a_N| \leq C|b_N|$ for some positive constant $C > 0$, and $A_N = o(B_N)$ means that $|a_N| \leq c_N|b_N|$ for some positive constants $c_N \rightarrow 0$ as $N \rightarrow \infty$. We also use the notation $a_N \sim b_N$ if $a_N = O(b_N)$ and $b_N = O(a_N)$. For any two vectors $v, u \in \mathbb{C}^n$, we define the inner product by $\langle v, u \rangle = v^* u$. For any matrix A , we denote by A^* as the transpose of A if A is a real matrix and the conjugate transpose if A is a complex matrix and we denote $\|A\|$ by its matrix norm when the dimension is fixed. In this paper, we usually write an $n \times n$ identity matrix $I_{n \times n}$ as 1 or I when there is no confusion about the dimension. We will also use $\sigma(H)$ to denote its spectrum for any square matrix H . And for any $M \times N$ rectangle matrix S we use $\sigma_i(S)$ to denote its i -th largest singular value.

2 Statistical applications

In this section, we will provide our estimators for (1.1) under the two regimes. We start with the case when u, v are sparse and then the rotation invariant estimation.

2.1 Sparse estimation

In the present application, we study denoising model (1.1), where S is sparse in the sense that the nonzero entries are assumed to be confined on a block. We assume that u_i, v_i are sparse and introduce the following notation to precisely describe the sparsity.

Definition 2.1 (Sparse vector). *For any vector $\nu \in \mathbb{R}^N$, ν is a sparse vector if there exists a subset $\mathbb{N}^* \subset \{1, 2, \dots, N\}$ with $|\mathbb{N}^*| = O(1)$, such that*

$$|\nu(i)| \leq \begin{cases} O(1), & i \in \mathbb{N}^*; \\ O(N^{-1/2}), & \text{otherwise.} \end{cases}$$

Denote

$$q = \underset{i}{\operatorname{argmin}} \{1 \leq i \leq K : \mu_i^2 \leq \lambda_+ + N^{-2/3}\}, \quad (2.1)$$

where λ_+ is defined in (1.5) and K is defined in (1.9). Therefore, q is defined as the index of the first extremal non-outlier eigenvalue. By the discussion of (1.13) and (1.14), when $d_i > c^{-1/4}$, the corresponding μ_i will converge to $p(d_i)$ defined in (1.11). It is easy to check that $p(d)$ is an increasing function of d , combining with the fact $p(c^{-1/4}) = \lambda_+$, we can conclude that there exists $q - 1$ outliers. Graphically, a phase transition will happen after μ_q . Figure 1 shows such a phenomenon.

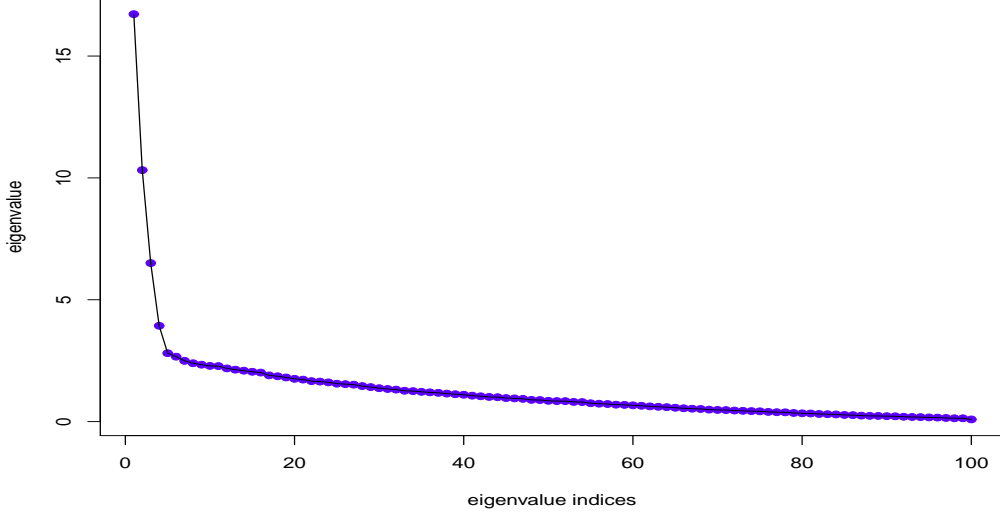


Figure 1: Eigenvalue phase transition. X is a 100×200 random Gaussian matrix. For S defined in (1.3), $r = 5$, with $d_1 = 4, d_2 = 3, d_3 = 2.5, d_4 = 1.5, d_5 = 0.1$. As $d_5 < 2^{-1/4} < d_4 < d_3 < d_2 < d_1$, we would expect four outliers. Hence, $q = 5$.

With the above notations, we provide the following *stepwise SVD* algorithm to recover S in (1.1). We aim to estimate the singular values and vectors respectively. As u_i, v_i are sparse, we need to find a submatrix of \tilde{S} by a suitable truncation. Instead of simply truncating the singular values [12, 26], we truncate the singular values and vectors simultaneously.

Algorithm 1 Stepwise SVD

- 1: Do SVD for $\tilde{S} = \sum_{i=1}^K \mu_i \tilde{u}_i \tilde{v}_i^*$, and do the initialization $\tilde{S}_1 = \tilde{S} = \sum t_i^1 \tilde{u}_i^1 (\tilde{v}_i^1)^*$.
- 2: **while** $1 \leq j < q$ **do**
- 3: $\hat{d}_j = p^{-1}((t_1^j)^2)$, where $p^{-1}(x)$ is the inverse of the function defined in (1.11).
- 4: Using two thresholdings $\alpha_{u_j} \gg \frac{1}{\sqrt{M}}$, $\alpha_{v_j} \gg \frac{1}{\sqrt{N}}$, denote

$$I_j := \{1 \leq k \leq M : |\tilde{u}_1^j(k)| \geq \alpha_{u_j}\}, \quad J_j := \{1 \leq k \leq N : |\tilde{v}_1^j(k)| \geq \alpha_{v_j}\}. \quad (2.2)$$

- 5: Do SVD for the block matrix $\tilde{S}_b = \tilde{S}[I_j, J_j] = \sum \rho_i u_i^j (v_i^j)^*$.
 - 6: Construct $\hat{u}_j = (0, u_i^j, 0)$, $\hat{v}_j = (0, v_i^j, 0)$ and keep the original indices of I_j, J_j .
 - 7: Let $\tilde{S}_{j+1} = \tilde{S}_j - \hat{d}_j \hat{u}_j \hat{v}_j^*$ and do SVD for $\tilde{S}_{j+1} = \sum t_i^{j+1} \tilde{u}_i^{j+1} (\tilde{v}_i^{j+1})^*$.
 - 8: Denote $\hat{S} = \sum_{k=1}^{q-1} \hat{d}_k \hat{u}_k \hat{v}_k^*$ as our estimator.
-

Algorithm 1 provides us a way to recover S stepwisely. We first estimate d_1, u_1, v_1 using the estimation $\hat{d}_1, \hat{u}_1, \hat{v}_1$, then continue estimating d_2, u_2, v_2 by analyzing $\tilde{S} - \hat{d}_1 \hat{u}_1 \hat{v}_1^*$. In each step, we only need to look at the largest singular value and its associated singular vector. It is notable that, we drop all the singular

values of \tilde{S} when they are below the level $\lambda_+ + N^{-2/3}$. This is due to the fact that, below this level, the noise will dominate the signal, we have nothing to estimate. Therefore, the shrinkage of singular values can be denoted as

$$\hat{d}_i = \mathbf{1}(\mu_i > \lambda_+ + N^{-2/3})p^{-1}(\mu_i), \quad (2.3)$$

where $p^{-1}(x)$ is the inverse of the function defined in (1.11). As we can see from (1.13), (2.3) outperforms the commonly used soft thresholding by simply denoting [9, 26]

$$\hat{d}_i = \mathbf{1}(\mu_i > \gamma)\mu_i,$$

especially when d_i is above the level $\lambda_+ + N^{-2/3}$.

Our methodology relies on truncating singular values and vectors together. As illustrated in (2.2), the thresholds α_u and α_v play the key roles in recovering the sparse structure of the singular vectors. It will be proved in Section 3 that any thresholds satisfying (2.2) should work when N is sufficiently large. In the finite sample framework (when N is not quite large), we will employ the unsupervised learning method, the K-means algorithm [14, Section 10.3.1] to stabilize the choices of the sparse structure in (2.2). The reason behind is, the entries in the singular vectors \tilde{u}_i, \tilde{v}_i can be classified into two categories: above and below the thresholds. We focus on the explanation for the right singular vectors. Let C_1, C_2 denote the sets of indices satisfying

$$C_1 \cup C_2 = \{1, 2, \dots, N\}, \quad C_1 \cap C_2 = \emptyset,$$

where C_1 contains the indices of the J_j in (2.2). Therefore, in practice, α_{v_j} is the entry of minimal absolute value in the class C_1 . Denote C_{1u}^j, C_{1v}^j as the subset of indices of $\tilde{u}_1^j, \tilde{v}_1^j$ respectively, where the minimal absolute value should satisfy

$$\min_{k \in C_{1u}^j} |\tilde{u}_1^j(k)| \gg \frac{1}{\sqrt{M}}, \quad \min_{k \in C_{1v}^j} |\tilde{v}_1^j(k)| \gg \frac{1}{\sqrt{N}}. \quad (2.4)$$

We now replace (2.2) with the following step:

- Do K-means clustering to partition $\tilde{u}_1^j, \tilde{v}_1^j$ into two classes, where we denote

$$I_j := \{1 \leq k \leq M : k \in C_{1u}^j\}, \quad J_j := \{1 \leq k \leq N : k \in C_{1v}^j\}, \quad (2.5)$$

where C_{1u}^j, C_{1v}^j satisfy (2.4).

In [26], the authors proposed another algorithm from a quite different perspective. They do not take the properties of the singular values and vectors of \tilde{S} into consideration. Instead, they use iterative thresholding on the rows of \tilde{S} to get an estimator. The algorithm is called *sparse SVD*. Their algorithm can be regarded as the extension of TSVD on the submatrix of \tilde{S} .

We use Table 1 to illustrate our simulation results. We consider two situations when $M = 300, N = 600$ and $M = 500, N = 1000$ respectively with different cases of sparsity. For each case, we perform 1000 Monte-Carlo simulations, recording the L^2 norm $\|S - \hat{S}\|_2$ defined in (2.6) and their standard deviation of the estimation. We compare the results of three algorithms, our stepwise SVD(SWSVD), the sparse SVD (SSVD) proposed by [26] and the truncated SVD(TSVD). For the generating of sparse vectors, we use the *RImagic* package in R and for the implementation of SSVD, we use the *ssvd* package in R which is contributed by the first author of [26].

From Table 1, we find that our method outperforms both the SSVD and TSVD in all the cases in the L^2 norm. Furthermore, the standard deviation is quite small, which implies that our estimation is quite stable.

In many biology applications, for example in the analysis of microarray gene expression data [18], we also attempt to recover the singular vectors to distinguish different types of gene patterns. Figure 2 is an example of the reconstruction of the left singular vector.

Remark 2.2. For the real data application, the noise level (i.e. the variance of q_{ij}) is usually unknown and we are required to estimate it. However, its information is embedded in the value of μ_q . Actually, when the variance of q_{ij} is σ^2 , then we have [3]

$$\mu_q \rightarrow \sigma^2 \lambda_+, \text{ a.s..}$$

Therefore, σ can be consistently estimated from $\hat{\sigma}$ defined by

$$\hat{\sigma} = \frac{\mu_q}{\lambda_+}.$$

	M=300			M=500		
	Sparsity	L^2 norm	Std	Sparsity	L^2 norm	Std
SWSVD	0.05	0.043	0.175	0.05	0.045	0.189
	0.1	0.614	0.178	0.1	0.6	0.16
	0.2	0.822	0.126	0.2	0.825	0.137
	0.45	1.1	0.114	0.45	1.09	0.09
SSVD	0.05	4.01	0.002	0.05	4.01	0.002
	0.1	4.01	0.004	0.1	4.02	0.002
	0.2	4.04	0.004	0.2	4.03	0.004
	0.45	4.06	0.005	0.45	4.08	0.004
TSVD	0.05	53.9	6.872	0.05	53.75	6.63
	0.1	53.72	6.63	0.1	53.38	6.71
	0.2	52.33	7.01	0.2	52.2	6.65
	0.45	51.043	2.49	0.45	52.4	4.3

Table 1: Comparison of the algorithms. We choose $r = 2, c = 2, d_1 = 7, d_2 = 4$ in (1.3), where the sparse vectors are generated by the *RImagic* library in R. The noise matrix X is Gaussian. In the table, sparsity is defined as the ratio of nonzero entries and the length of the vector and we assume that $u_i, v_i, i = 1, 2$ have the same sparsity. We highlight the smallest L^2 norm.

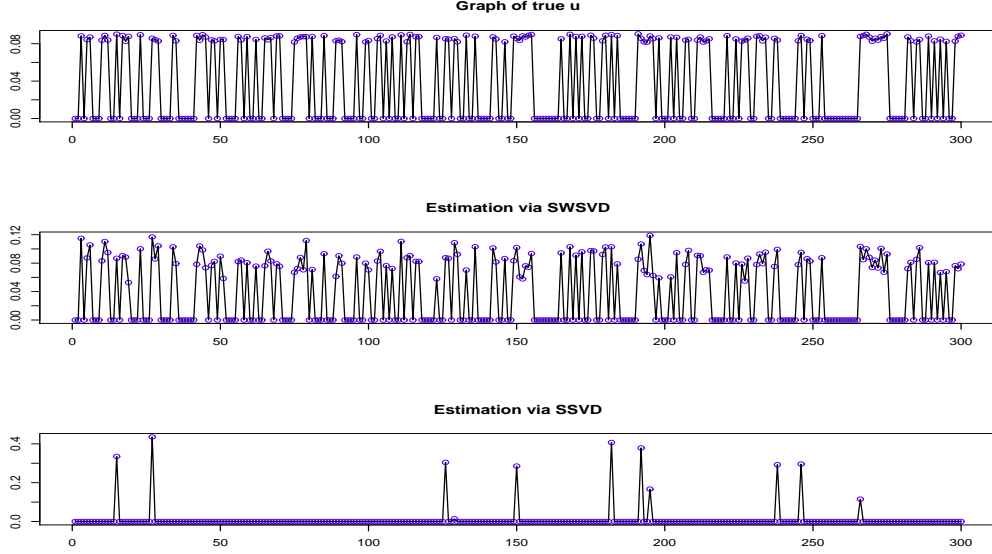


Figure 2: Estimation of the left singular vector. X is a 300×600 Gaussian matrix. We let $r = 1, d = 3$ in (1.3) and the number of non-zero entries of u to be 135. The top panel is the true singular vector and the rest two are the estimations from stepwise SVD and sparse SVD respectively. We find that the estimation from sparse SVD can be misleading when the singular vector is not very sparse, but our method still captures the sparsity structure.

2.2 Rotation invariant estimation

This section is devoted to recover S in (1.1) assuming that no prior information about S is available. In this regime, we will consider the rotation invariant estimator satisfying (1.19). We will attempt to construct a consistent RIE only relying on the given observation \hat{S} . We conclude from [6] that any RIE shares the same singular vectors as \hat{S} . To construct the optimal estimator, we use the Frobenius norm as our loss function. Denote $\hat{S} = \Xi(\hat{S})$, we have

$$\|S - \hat{S}\|_2^2 = \text{Tr}(S - \hat{S})(S - \hat{S})^*. \quad (2.6)$$

Therefore, the form of the RIE can be written in the following way

$$\hat{S} = \underset{H \in \mathcal{M}(\tilde{U}, \tilde{V})}{\operatorname{argmin}} \|H - S\|_2, \quad (2.7)$$

where $\mathcal{M}(\tilde{U}, \tilde{V})$ is the class of $M \times N$ matrices whose left singular vectors are \tilde{U} and right singular vectors are \tilde{V} . Suppose

$$\hat{S} = \sum_{i=1}^K \eta_k \tilde{u}_k \tilde{v}_k^*, \quad (2.8)$$

then by an elementary computation, we find

$$\begin{aligned} \|S - \hat{S}\|_2^2 &= \sum_{k=1}^r (d_k^2 + \eta_k^2) - 2 \sum_{k=1}^r d_k \eta_k \mu_{kk} \nu_{kk} \\ &\quad + \sum_{k=r+1}^K \eta_k^2 - 2 \sum_{k_1 \neq k_2}^r d_{k_1} \eta_{k_2} \mu_{k_1 k_2} \nu_{k_1 k_2} - 2 \sum_{k_1=r+1}^K \sum_{k_2=1}^r \eta_{k_1} d_{k_2} \mu_{k_2 k_1} \nu_{k_2 k_1}, \end{aligned} \quad (2.9)$$

where

$$\mu_{k_1 k} = \langle u_{k_1}, \tilde{u}_k \rangle, \quad \nu_{k_1 k} = \langle v_{k_1}, \tilde{v}_k \rangle.$$

Therefore, it is easy to check that \hat{S} is optimal if

$$\eta_k = \langle \tilde{u}_k, S \tilde{v}_k \rangle = \sum_{k_1=1}^r d_{k_1} \mu_{k_1 k} \nu_{k_1 k}, \quad k = 1, \dots, K. \quad (2.10)$$

In the present paper, we will use the following estimator for η_k and prove its consistency in Section 3. The estimator reads as

$$\hat{\eta}_k = \begin{cases} \hat{d}_k a_1(\hat{d}_k) a_2(\hat{d}_k), & k \leq q-1; \\ 0, & k \geq q. \end{cases} \quad (2.11)$$

where $\hat{d}_k = p^{-1}(\mu_k)$ and $p^{-1}(\mu_k)$ is value of the inverse function defined in (1.11), $a_1(x), a_2(x)$ are defined as

$$a_1(x) = \frac{x^4 - c^{-1}}{x^2(x^2 + c^{-1})}, \quad a_2(x) = \frac{x^4 - c^{-1}}{x^2(x^2 + 1)}, \quad (2.12)$$

and q is defined in (2.1). Figure 3 are two examples of estimations of η_k .

Our method provides better estimation compared to the TSVD. Figure 4 records the relative improvement in average loss (RIAL) compared to the TSVD. The RIAL is defined as

$$\text{RIAL}(N) = 1 - \frac{\mathbb{E} \|\hat{S} - S\|_2}{\mathbb{E} \|S_N - S\|_2}, \quad (2.13)$$

where S_N is the TSVD estimation of \tilde{S} and \hat{S} is the RIE of S . By construction, the RIAL of the TSVD is 0, meaning no improvement.

Remark 2.3. In [12], Donoho and Gavish get similar results from the perspective of optimal shrinkage. However, they need two more assumptions: (1). they drop the last two error terms in (2.9) by assuming they are small enough (see Lemma 4 in their paper); (2) their estimators are assumed to be conservative (see Definition 3 in their paper), where

$$\hat{\eta}_k = 0, \quad k \geq q.$$

Their methodologies actually quite rely on these two assumptions to make the error terms vanish. However, we find that the estimator defined in (2.11) is still consistent even without these two assumptions.

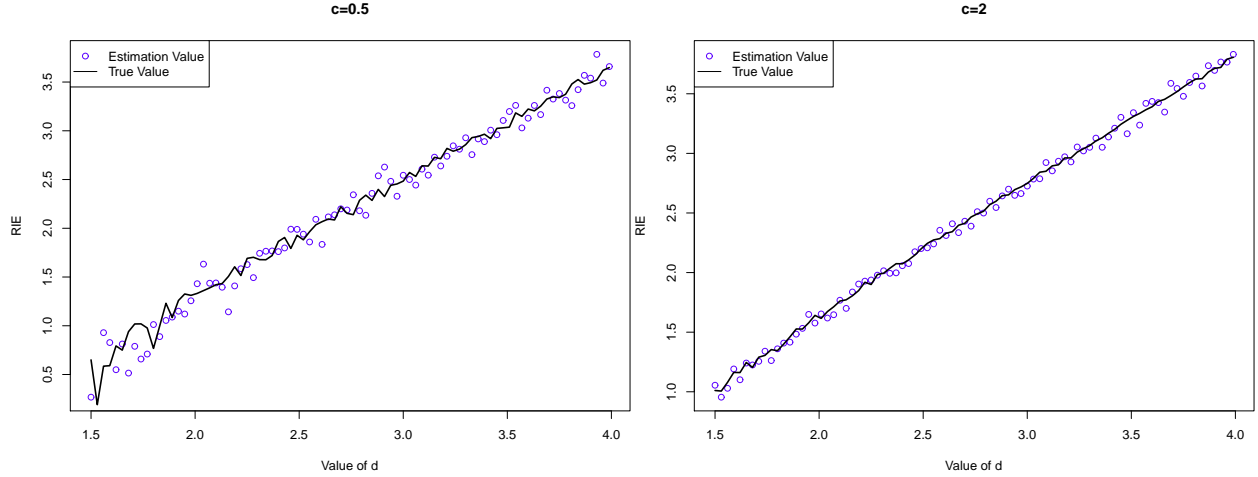


Figure 3: RIE. We choose $r = 1$ and $M = 300$ for (1.1). We estimate η_1 using the estimator (2.11) for $c = 0.5, 2$ respectively with different values of d . The entries of X are Gaussian random variables and the singular vectors satisfy the exponential distribution with rate 1.

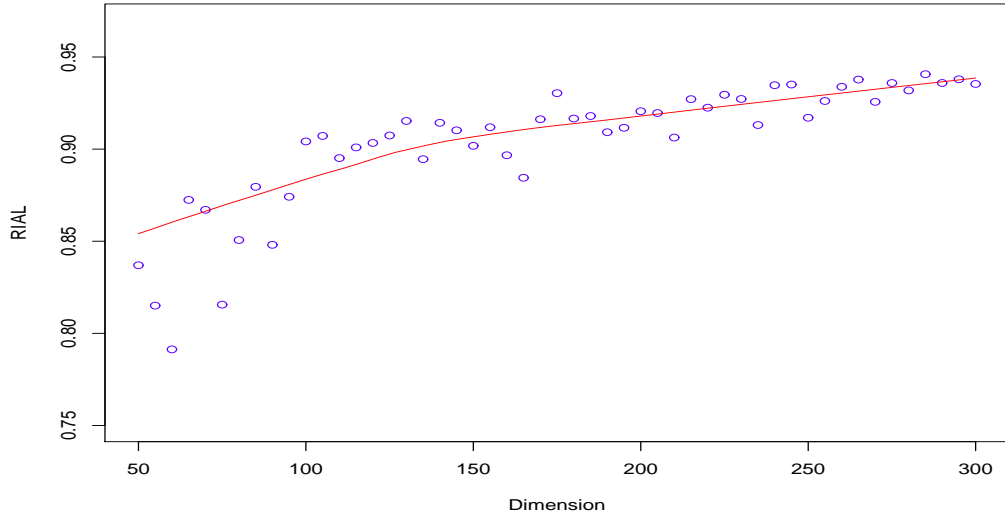


Figure 4: RIE compared to TSVD. We choose $r = 1, d = 4, c = 2$ in (1.1). X is a random Gaussian matrix and the entries of the singular vectors satisfy the exponential distribution with rate 1. We perform 1000 Monte-Carlo simulations for each M to simulate the RIAL defined in (2.13). The red line indicates the increasing trend as M increases.

3 Main results

In this section, we give the main results of this paper. We first introduce the following definition, which is [5, Definition 2.1]. It provides a way of making precise statement of the form that A is bounded by B up to N^{ϵ_1} with high probability greater than $1 - N^{-D_1}$.

Definition 3.1 (Stochastic domination). *Consider the following two families of nonnegative random variables,*

$$\xi = (\xi^N(u) : N \in \mathbb{N}, u \in U^{(N)}), \quad \zeta = (\zeta^{(N)}(u) : N \in \mathbb{N}, u \in U^{(N)}),$$

where $U^{(N)}$ is a possibly N -dependent parameter set. We say that ξ is stochastically dominated by ζ , uni-

formly in u , if for all ϵ small enough and D large enough, we have

$$\sup_{u \in U(N)} \mathbb{P}[\xi^N(u) > N^\epsilon \zeta^N(u)] \leq N^{-D},$$

for large enough $N \geq N_0(\epsilon, D)$. $U^{(N)}$ usually stands for the matrix indices, some deterministic vectors or the domain of spectral parameter z . If ξ is stochastically dominated by ζ uniformly in u , we write as $\xi \prec \zeta$. Furthermore, an N -dependent event Ξ holds with high probability, if $1 - \mathbf{1}(\Xi) \prec 0$.

Throughout the paper, we will use ϵ_1 for the small constant and D_1 for the large constant whenever the stochastic domination is satisfied. Denote \mathcal{R} as the set of $d_i, i = 1, 2, \dots, r$ and \mathcal{O} as a subset of \mathcal{R} by

$$\mathcal{O} := \{d_i : d_i \geq c^{-1/4} + N^{-1/3+\epsilon_0}\}, \quad \epsilon_0 \gg \epsilon_1, \quad (3.1)$$

and

$$k^+ = |\mathcal{O}|. \quad (3.2)$$

Remark 3.2. Our results can be extended to a more general domain by denoting

$$\mathcal{O}' := \{d_i : d_i \geq c^{-1/4} + N^{-1/3}\}.$$

The proofs still hold true with some minor changes except we need to discuss the case when $d_i \in (c^{-1/4} + N^{-1/3}, c^{-1/4} + N^{-1/3+\epsilon_0})$. We will not pursue this generalization. For more details, we refer to [5].

For any subset $A \subset \mathcal{O}$, we define the projections on the left and right singular subspace of \tilde{S} by

$$\mathbf{P}_l := \sum_{i \in A} \tilde{u}_i \tilde{u}_i^*, \quad \mathbf{P}_r := \sum_{j \in A} \tilde{v}_j \tilde{v}_j^*. \quad (3.3)$$

As discussed in (1.16), we need the non-overlapping condition, which was firstly introduced in [5, (5.3)].

Definition 3.3. The non-overlapping condition is written as

$$\nu_i(A) \geq (d_i - c^{-1/4})^{-1/2} N^{-1/2+\epsilon_0}, \quad (3.4)$$

where ϵ_0 is defined in (3.1) and for $i \in [1, M] \cap \mathbb{Z}$, ν_i is defined by

$$\nu_i \equiv \nu_i(A) := \begin{cases} \min_{j \notin A} |d_i - d_j|, & \text{if } i \in A, \\ \min_{j \in A} |d_i - d_j|, & \text{if } i \notin A. \end{cases} \quad (3.5)$$

With the above preparations, we now state our main results. The following theorem characterizes the local behavior of the singular values of \tilde{S} .

Theorem 3.4 (Location of singular values). *For all $i \in \mathcal{O}$, $i = 1, 2, \dots, k^+$, where k^+ is defined in (3.2), there exists some large constant $C > 1, C\epsilon_1 < \epsilon_0$, with $1 - N^{-D_1}$ probability, we have*

$$|\mu_i - p(d_i)| \leq N^{-1/2+C\epsilon_0} (d_i - c^{-1/4})^{1/2}, \quad (3.6)$$

where $p(d_i)$ is defined in (1.11). Moreover, for $j \in \mathcal{R}/\mathcal{O}, j = k^+ + 1, \dots, r$, with $1 - N^{-D_1}$ probability, we have

$$|\mu_j - \lambda_+| \leq N^{-2/3+C\epsilon_0}, \quad (3.7)$$

where λ_+ is defined in (1.5).

The above theorem gives precise location of the outlier singular values and the extremal non-outlier singular values. For the outliers, they will locate around their classical locations $p(d_i)$ and for the non-outliers, they will locate around λ_+ . However, (3.7) can be easily extended to a more general framework. Instead of considering λ_+ , we can locate μ_j around the eigenvalues of XX^* , which is the phenomenon of *eigenvalue sticking*. Due to our application motivation, we will not follow this direction and the details can be referred to [5, Theorem 2.7].

The results of the singular vectors are given by the following theorem.

Theorem 3.5 (Delocalization of singular vectors). *For all $i, j = 1, 2, \dots, r$, there exists some constant $C > 0$, under the assumption (3.4), with $1 - N^{-D_1}$ probability, we have*

$$| \langle u_i, \mathbf{P}_l u_j \rangle - \delta_{ij} \mathbf{1}(i \in A) a_1(d_i) | \leq N^{\epsilon_1} R(i, j, A, N), \quad (3.8)$$

$$| \langle v_i, \mathbf{P}_r v_j \rangle - \delta_{ij} \mathbf{1}(i \in A) a_2(d_i) | \leq N^{\epsilon_1} R(i, j, A, N), \quad (3.9)$$

where $a_1(x), a_2(x)$ are defined in (2.12) and $R(i, j, A, N)$ is the error term depending on i, j, A, N and defined as

$$R(i, j, A, N) := N^{-1/2} \left[\frac{\mathbf{1}(i \in A, j \in A)}{(d_i - c^{-1/4})^{1/2} + (d_j - c^{-1/4})^{1/2}} + \mathbf{1}(i \in A, j \notin A) \frac{(d_i - c^{-1/4})^{1/2}}{|d_i - d_j|} \right. \\ \left. + \mathbf{1}(i \notin A, j \in A) \frac{(d_j - c^{-1/4})^{1/2}}{|d_i - d_j|} \right] + N^{-1} \left[\left(\frac{1}{\nu_i} + \frac{\mathbf{1}(i \in A)}{|d_i - c^{-1/4}|} \right) \left(\frac{1}{\nu_j} + \frac{\mathbf{1}(j \in A)}{|d_j - c^{-1/4}|} \right) \right].$$

Moreover, for $j = k^+ + 1, \dots, K$, denote $\kappa_j := |\mu_j - \lambda_+|$, with $1 - N^{-D_1}$ probability, we have

$$| \langle u_i, \tilde{u}_j \rangle | \leq \frac{N^{C\epsilon_0}}{N((d_i - c^{-1/4})^2 + \kappa_j)}, \quad i = 1, 2, \dots, r, \quad (3.10)$$

and

$$| \langle v_i, \tilde{v}_j \rangle |^2 \leq \frac{N^{C\epsilon_0}}{N((d_i - c^{-1/4})^2 + \kappa_j)}, \quad i = 1, 2, \dots, r. \quad (3.11)$$

Next we will give some examples to illustrate our results.

Example 3.6. (1). Consider the right singular vectors and let $A = \{i\}$, we have

$$| \langle v_i, \tilde{v}_i \rangle - a_2(d_i) | \leq N^{\epsilon_1} \left[\frac{1}{N^{1/2}(d_i - c^{-1/4})^{1/2}} + \frac{1}{N\nu_i(d_i - c^{-1/4})} \right].$$

This implies that, the cone concentration of the singular vector holds if $i \in \mathcal{O}$ and the non-overlapping condition (3.4) holds true. Furthermore, if d_i is well-separated from both the critical point $c^{-1/4}$ and the other outliers, the error bound is of order $\frac{1}{\sqrt{N}}$.

(2). Let $A = \{i\}$ and for $1 \leq j \neq i \leq r$, we have

$$| \langle v_j, \tilde{v}_i \rangle | \leq N^{2\epsilon_1} \frac{1}{N(d_i - d_j)^2}.$$

Hence, if $|d_i - d_j| = O(1)$, then \tilde{v}_i will be completely delocalized in any direction orthogonal to v_j .

(3). If $i \notin \mathcal{O}$, then we have

$$| \langle v_j, \tilde{u}_i \rangle | \leq \frac{N^{C\epsilon_0}}{N((d_j - c^{-1/4})^2 + \kappa_i)}.$$

Hence, when $|d_j - c^{-1/4}| = O(1)$ or $\kappa_i = O(1)$, \tilde{u}_i will be completely delocalized in the direction of v_j . The first case reads as μ_j is an outlier and the second case as that μ_i is in the bulk of the spectrum of $\tilde{S}\tilde{S}^*$.

Once armed with Theorem 3.4 and Theorem 3.5, we can derive the following consistency property of our estimators defined in Section 2.

Theorem 3.7 (Consistency of the estimators). *For the matrix denosing model (1.1), under the assumption (3.4),*

(1). *With the prior information that u_i, v_i are sparse in the sense of Definition 2.1, we have that*

- I_i, J_i defined in (2.2) are consistent estimators of the indices of the nonzero entries of u_i, v_i by choosing $\alpha_{u_i} \geq N^{-1/2+\delta_1}, \alpha_{v_i} \geq N^{-1/2+\delta_1}, \epsilon_1 \ll \delta_1 < \frac{1}{2}$ if $d_i \in \mathcal{O}$.

- For the estimator \hat{S} from the **Algorithm 1**, with $1 - N^{-D_1}$ probability, we have

$$\|\hat{S} - S\|_2^2 \rightarrow \sum_{k=k^*+1}^r d_i^2.$$

Hence, if $\mathcal{R} = \mathcal{O}$, \hat{S} will be a consistent estimator for S .

(2). For the rotation invariant estimator defined in (2.8) and (2.10), $\hat{\eta}_k$ is a consistent estimator for η_k , $k = 1, \dots, K$.

Proof. We start with the proof of (1). For the first part, due to similarity, we only prove for the right singular vectors. As our algorithm is a stepwise procedure, without loss of generality, we only consider the rank-one case, and the rank r case can be proved by induction. Denote the indices set of nonzero entries of v by J , and denote μ as the largest eigenvalue of $\tilde{S}^* \tilde{S}$ and \tilde{v} as the corresponding eigenvector. We start to show that there exists some constant $\delta > 0$, such that $|\tilde{v}(k)| \geq CN^{-1/2+\delta}$, $k \in J$ with $1 - N^{-D_1}$ probability. Suppose that there exists a $k^* \in J$, such that $|\tilde{v}(k^*)| \leq N^{-1/2+\delta}$ where $0 < \delta < \frac{1}{2}$, by definition,

$$(\tilde{S}^* \tilde{S} \tilde{v})(k^*) = \mu \tilde{v}(k^*), \quad (\tilde{S}^* \tilde{S} \tilde{v})(k_1) = \mu \tilde{v}(k_1), \quad k_1 \in J, \quad k_1 \neq k^*,$$

which reads as

$$\begin{aligned} \tilde{v}(k^*) \sum_{p=1}^M (x_{pk^*})^2 + \sum_{k \neq k^* \in J} \left[\tilde{v}(k) \sum_{p=1}^M x_{pk} x_{pk^*} \right] + \sum_{t \notin J} \left[\tilde{v}(t) \sum_{p=1}^M x_{pt} x_{pk^*} \right] &= \mu \tilde{v}(k^*), \\ \tilde{v}(k_1) \sum_{p=1}^M (x_{pk_1})^2 + \sum_{k \neq k_1 \in J} \left[\tilde{v}(k) \sum_{p=1}^M x_{pk} x_{pk_1} \right] + \sum_{t \notin J} \left[\tilde{v}(t) \sum_{p=1}^M x_{pt} x_{pk_1} \right] &= \mu \tilde{v}(k_1). \end{aligned} \quad (3.12)$$

It is easy to verify that with $1 - N^{-D_1}$ probability, when $k_1 \in J$, $k_1 \neq k^*$,

$$\left| \sum_{t \notin J} \left[\tilde{v}(t) \sum_{p=1}^M x_{pt} x_{pk^*} \right] - \sum_{t \notin J} \left[\tilde{v}(t) \sum_{p=1}^M x_{pt} x_{pk_1} \right] \right| \leq N^{-\frac{1}{2}+\epsilon_1}. \quad (3.13)$$

(3.12) implies that $\left| \sum_{t \notin J} \left[\tilde{v}(t) \sum_{p=1}^M x_{pt} x_{pk^*} \right] \right| \leq N^{-1/2+\delta}$. By (3.13), this immediately yields that $|\tilde{v}(k_1)| \leq N^{-1/2+\delta}$, $k_1 \in J$. Similarly, we have

$$|\tilde{v}(k)| \leq N^{-1/2+\delta}, \quad k \in J,$$

which implies that $|\langle \tilde{v}, v \rangle| \leq N^{-1/2+\delta}$. This is a contradiction by Theorem 3.5. Denote X_J as the minor of X by deleting the i -th columns where $i \in J$ and \tilde{v}_J as the subvector of \tilde{v} by deleting the entries with indices in J . To complete the proof, we next show that with $1 - N^{-D_1}$ probability,

$$|\tilde{v}_J(k)| \leq N^{-1/2+\epsilon_1}.$$

By definition, we have

$$\tilde{S}^* \tilde{S} \tilde{v} = X^* X \tilde{v} + X^* S \tilde{v} + S^* X \tilde{v} + S^* S \tilde{v} = \mu \tilde{v}.$$

For the k -th entry, where $k \in J^c \cap \{1, \dots, N\}$, we have

$$\mu \tilde{v}(k) = (X^* X \tilde{v})(k) + (X^* S \tilde{v})(k).$$

As $|J| = O(1)$, with $1 - N^{-D_1}$ probability, we have

$$\mu \tilde{v}_J(k) = (X_J^* X_J \tilde{v}_J)(k) + O(N^{-1/2+\epsilon_1}).$$

By a similar discussion to (3.12) and (3.13), if there exists a $k_0 \in J^c$ such that for some constant $\delta > 0$

$$|\tilde{v}_J(k_0)| \geq N^{-1/2+\delta}, \quad (3.14)$$

then (3.14) holds for all $k \in J^c \cap \{1, 2, \dots, N\}$. This yields that

$$X_J^* X_J \tilde{v}_J \rightarrow \mu \tilde{v}_J, \text{ pointwisely.}$$

As $\|\tilde{v}_J\| \leq 1$, we have that $\frac{1}{\|\tilde{v}_J\|} \tilde{v}_J^* X_J^* X_J \tilde{v}_J \rightarrow \mu$, which is a contradiction, as $X_J^* X_J$ should satisfy the MP law. Hence, we finish the proof of the first part. For the second part, by the results of the first part, the assumption of x_{ij} in (1.2), r is finite and Theorem 3.4, with $1 - N^{-D_1}$ probability, we have

$$\|\hat{S} - S\|_2^2 \rightarrow \left\| \sum_{j=k_++1}^r d_j u_j v_j^* \right\|_2^2 = \sum_{j=k_++1}^r d_j^2.$$

Therefore, if $k^+ = r$, \hat{S} is a consistent estimator for S . Next we prove (2). By (2.10), we have

$$\eta_k = \sum_{k_1=1}^{k^+} d_{k_1} \mu_{k_1 k} \nu_{k_1 k} + \sum_{k_1=k^++1}^r d_{k_1} \mu_{k_1 k} \nu_{k_1 k}.$$

Recall (2.12), by Theorem 3.5, when $k \leq k^+$, we have

$$\eta_k = d_k a_1(d_k) a_2(d_k) + o(1).$$

By Theorem 3.4, we have that $q - 1$ is a consistent estimator of k^+ and

$$\hat{d}_k = d_k + o(1).$$

Therefore, we have conclude that $\hat{\eta}_k$ is an consistent estimator when $k \leq k^+$. By (3.10) and (3.11), when $k > k^+$, we have

$$\eta_k = o(1),$$

which implies that $\hat{\eta}_k$ is also a consistent estimator when $k > k^+$. This concludes our proof of (2). \square

4 Notations and basic tools

In this section, we introduce some notations and basic tools which will be used in this paper. Recall that the empirical spectral distribution(ESD) of an $n \times n$ symmetric matrix H is defined as

$$F_H^{(n)}(\lambda) := \frac{1}{n} \sum_{i=1}^n \mathbf{1}_{\{\lambda_i(H) \leq \lambda\}}.$$

We also define the typical domain for $z = E + i\eta$ by

$$\mathbf{D}(\tau) = \{z \in \mathbb{C}^+ : |E| \leq \tau^{-1}, N^{-1+\tau} \leq \eta \leq \tau^{-1}, |z| \geq \tau\}, \quad (4.1)$$

where $\tau \in (0, 1)$ is small constant. We further assume that $\tau < c_N < \tau^{-1}$, where c_N is defined in (1.4).

Definition 4.1 (Stieltjes transform). *The Stieltjes transform of the ESD of $X^* X$ is given by*

$$m_2(z) \equiv m_2^{(N)}(z) := \int \frac{1}{x - z} dF_{X^* X}^{(N)}(x) = \frac{1}{N} \sum_{i=1}^N (\mathcal{G}_2)_{ii}(z) = \frac{1}{N} \text{Tr } \mathcal{G}_2(z), \quad (4.2)$$

where $\mathcal{G}_2(z)$ is defined in (1.6). Similarly, we can also define $m_1(z) \equiv m_1^{(M)}(z) := M^{-1} \text{Tr } \mathcal{G}_1(z)$.

Remark 4.2. Since the nonzero eigenvalues of XX^* and X^*X are identical and XX^* has $M - N$ more (or $N - M$ less) zero eigenvalues, we have

$$F_{XX^*}^{(M)} = c_N F_{X^* X}^{(N)} + (1 - c_N) \mathbf{1}_{[0, \infty)}, \quad (4.3)$$

and

$$m_1(z) = -\frac{1 - c_N}{z} + c_N m_2(z). \quad (4.4)$$

Denote

$$m_{1c}(z) := \lim_{N \rightarrow \infty} m_1(z), \quad m_{2c}(z) := \lim_{N \rightarrow \infty} m_2(z), \quad (4.5)$$

be the Stieltjes transforms of limiting spectral distribution of $m_1(z)$, $m_2(z)$. By (4.4), we have

$$m_{1c}(z) = \frac{c-1}{z} + dm_{2c}(z). \quad (4.6)$$

Definition 4.3 (Marchenko-Pastur law). *For X satisfying (1.2), under the assumption (1.4), the ESD of XX^* converges weakly to the Marchenko-Pastur (MP) law as $N \rightarrow \infty$ [19]:*

$$\rho_{1c}(x)dx = \frac{c}{2\pi} \frac{\sqrt{(\lambda_+ - x)(x - \lambda_-)}}{x} dx, \quad \lambda_{\pm} = (1 \pm c^{-\frac{1}{2}})^2. \quad (4.7)$$

Then $m_1(z)$ converges to the Stieltjes transform of the MP law $m_{1c}(z)$, which satisfies the following self-consistent equation (see (1.2) of [22])

$$m_{1c}(z) + \frac{1}{z - (1 - c^{-1}) + zc^{-1}m_{1c}(z)} = 0, \quad \text{Im } m_{1c}(z) \geq 0 \text{ for } z \in \mathbb{C}^+, \quad (4.8)$$

and has the closed form expression

$$m_{1c}(z) = \frac{1 - c^{-1} - z + i\sqrt{(\lambda_+ - z)(z - \lambda_-)}}{2zc^{-1}}, \quad (4.9)$$

where the square root denotes the complex square root with a branch cut on the negative real axis.

Remark 4.4. From (4.4), it is easy to see that $m_2(z)$ converges to $m_{2c}(z)$ as $N \rightarrow \infty$, where

$$m_{2c}(z) = \frac{c^{-1} - 1}{z} + c^{-1}m_{1c}(z) = \frac{c^{-1} - 1 - z + i\sqrt{(\lambda_+ - z)(z - \lambda_-)}}{2z}. \quad (4.10)$$

Moreover, it is easy to verify that $m_{2c}(z)$ satisfies the self-consistent equation (see (1.4) of [22])

$$m_{2c}(z) + \frac{1}{z + (1 - c^{-1}) + zm_{2c}(z)} = 0. \quad (4.11)$$

Correspondingly, we denote the probability density function for the limiting ESD of X^*X as

$$\rho_{2c}(x) := \frac{1}{2\pi} \frac{\sqrt{(\lambda_+ - x)(x - \lambda_-)}}{x}. \quad (4.12)$$

As we have discussed in (1.7), we will use the following linearizing block matrix. For $z \in \mathbb{C}^+$, we define the $(N + M) \times (N + M)$ self-adjoint matrices

$$G \equiv G(X, z) := \begin{pmatrix} -zI & z^{1/2}X \\ z^{1/2}X^* & -zI \end{pmatrix}. \quad (4.13)$$

By Schur's complement [15], it is easy to check that

$$G = \begin{pmatrix} \mathcal{G}_1(z) & z^{-1/2}\mathcal{G}_1(z)X \\ z^{-1/2}X^*\mathcal{G}_1(z) & z^{-1}X^*\mathcal{G}_1(z)X - z^{-1}I \end{pmatrix} = \begin{pmatrix} z^{-1}X\mathcal{G}_2(z)X^* - z^{-1}I & z^{-1/2}X\mathcal{G}_2(z) \\ z^{-1/2}\mathcal{G}_2(z)X^* & \mathcal{G}_2(z) \end{pmatrix}, \quad (4.14)$$

for $\mathcal{G}_{1,2}$ defined in (1.6). Thus a control of G yields directly a control of $(XX^* - z)^{-1}$ and $(X^*X - z)^{-1}$. Denote the index sets

$$\mathcal{I}_1 := \{1, \dots, M\}, \quad \mathcal{I}_2 := \{M + 1, \dots, M + N\}, \quad \mathcal{I} := \mathcal{I}_1 \cup \mathcal{I}_2.$$

Then we have

$$m_1(z) = \frac{1}{M} \sum_{i \in \mathcal{I}_1} G_{ii}, \quad m_2(z) = \frac{1}{N} \sum_{\mu \in \mathcal{I}_2} G_{\mu\mu}.$$

Similarly, we denote $\tilde{G}(z) = (\tilde{H} - z)^{-1}$, where \tilde{H} is defined in (1.7). Next we introduce the spectral decomposition of \tilde{G} . By (4.14), we have

$$\tilde{G}(z) = \sum_{k=1}^K \frac{1}{\mu_k - z} \begin{pmatrix} \tilde{u}_k^* & z^{-1/2} \sqrt{\mu_k} \tilde{u}_k \tilde{v}_k^* \\ z^{-1/2} \sqrt{\mu_k} \tilde{v}_k \tilde{u}_k^* & \tilde{v}_k \tilde{v}_k^* \end{pmatrix}. \quad (4.15)$$

where $\tilde{u}_k, \tilde{v}_k, \sqrt{\mu_k}, k = 1, 2, \dots, K$ are the left singular vectors, right singular vectors and singular values of \tilde{S} respectively. Denote

$$\Psi(z) := \sqrt{\frac{\text{Im } m_{2c}(z)}{N\eta}} + \frac{1}{N\eta}, \quad \underline{\Sigma} := \begin{pmatrix} z^{-1/2} & 0 \\ 0 & I \end{pmatrix}. \quad (4.16)$$

Definition 4.5 (Deterministic convergent limit of G). *For $z \in \mathbb{C}^+$, we define the $(N+M) \times (N+M)$ matrix*

$$\Pi(z) := \begin{pmatrix} -z^{-1}(1 + m_{2c}(z))^{-1} & 0 \\ 0 & m_{2c}(z) \end{pmatrix} = \begin{pmatrix} m_{1c}(z) & 0 \\ 0 & m_{2c}(z) \end{pmatrix}. \quad (4.17)$$

It has been shown that [7, 15], with high probability, $G(z)$ converges to $\Pi(z)$.

As we have seen in (1.11), the function $p(d)$ plays a key role in describing the convergent limit of the singular values of \tilde{S} . An elementary computation yields that $p(d)$ attains its global minimum when $d = c^{-1/4}$ and $p(c^{-1/4}) = \lambda_+$, where λ_+ is defined in (1.5). Recall (1.10), to precisely locate the singular values of \tilde{S} , we will consider

$$T^s(x) := \prod_{i=1}^s (zm_{1c}(x)m_{2c}(x) - d_i^{-2}), \quad (4.18)$$

where $m_{1c}(z), m_{2c}(z)$ are defined in (4.9) and (4.10) respectively. Figure 5 is an example of $T^s(x)$.

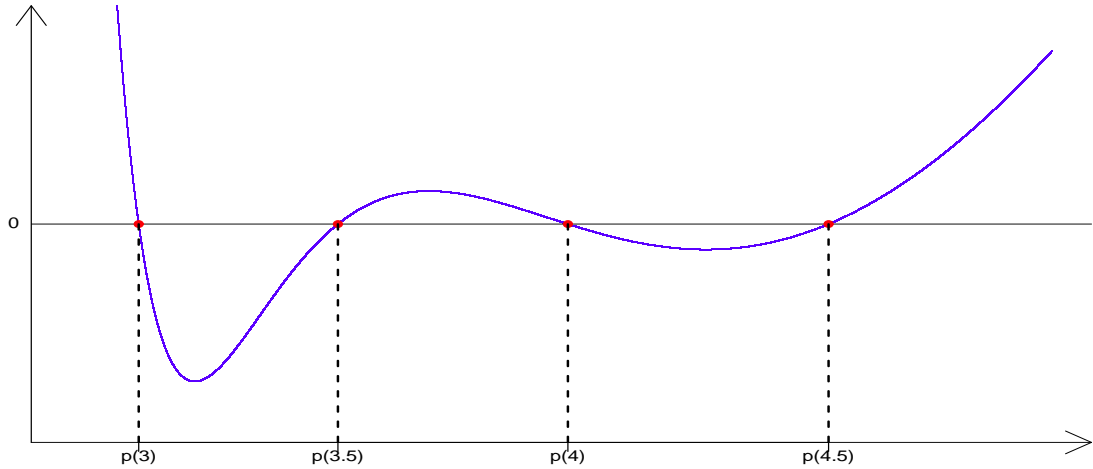


Figure 5: Graph for $T^4(x)$. We choose $c = 2, s = 4$ and the corresponding d_i to be 3, 3.5, 4, 4.5 respectively.

By (4.8) and (4.11), when $x \geq \lambda_+$, we have

$$xm_{1c}(x)m_{2c}(x) = \frac{x - (1 + c^{-1}) - \sqrt{(x - \lambda_+)(x - \lambda_-)}}{2c^{-1}} = \frac{x - (1 + c^{-1}) - \sqrt{(x + c^{-1} - 1)^2 - 4c^{-1}x}}{2c^{-1}}. \quad (4.19)$$

We next collect the preliminary results of the properties of $T^s(x)$.

Lemma 4.6. *Suppose $d_1 > d_2 > \dots > d_s > c^{-1/4}$, then we have*

(1). There exist s solutions of $T^s(x) = 0$ and they are $p_i := p(d_i), i = 1, 2, \dots, s$, respectively, write

$$T^s(p_i) = 0. \quad (4.20)$$

(2). There are $s - 1$ critical points x_1, \dots, x_{s-1} of $T^s(x)$ and $x_i \in (p_{i+1}, p_i), i = 1, 2, \dots, s - 1$.

(3). Denote

$$\mathcal{T}(x) := xm_{1c}(x)m_{2c}(x), \quad (4.21)$$

then $\mathcal{T}(x)$ is a strictly monotone decreasing function when $x > \lambda_+$.

Proof. (4.20) is from an elementary calculation and (2) is due to the mean value theorem. For the proof of (3), $\forall x > y > \lambda_+$, we have

$$xm_{1c}(x)m_{2c}(x) - ym_{1c}(y)m_{2c}(y) = \frac{x - y - (g(x) - g(y))}{2c^{-1}},$$

where $g(t) := \sqrt{(t + c^{-1} - 1)^2 - 4c^{-1}t}$. When $t > \lambda_+$, we have

$$g'(t) = \frac{t - (c^{-1} + 1)}{\sqrt{t^2 + (c^{-1} - 1)^2 - 2t(c^{-1} + 1)}} > \frac{t - (c^{-1} + 1)}{\sqrt{t^2 + (c^{-1} + 1)^2 - 2t(c^{-1} + 1)}} = 1,$$

where we need $t > \lambda_+$ to ensure the positiveness of $g(t)$. Hence, by the mean value theorem, we conclude the proof. \square

Remark 4.7. By Lemma 4.6, we find that, when s is an even number, $T^s(x)$ is positive when $x < p(d_s)$ and $x > p(d_1)$ and alternating on the intervals $(p(d_{i+1}), p(d_i)), i = 1, 2, \dots, s - 1$. And when s is an odd number, $T^s(x)$ is positive when $x < d_s$ and negative when $x > d_1$ and alternating on the other intervals.

For $z \in \mathbf{D}(\tau)$ defined in (4.1), denote

$$\kappa := ||E| - \lambda_+|. \quad (4.22)$$

The following lemma summarizes the basic properties of $m_{1c}(z)$ and $m_{2c}(z)$, its proofs are based on elementary calculations of (4.9) and (4.10), which can be found in [4, Lemma 3.3] and [5, Lemma 3.6].

Lemma 4.8.

$$|m_{2c}(z)| \sim c^{1/2}, \quad |1 - m_{2c}^2(z)| \sim \sqrt{\kappa + \eta},$$

and

$$\operatorname{Im} m_{1c}(z) \sim \operatorname{Im} m_{2c}(z), \quad \operatorname{Im} m_{2c}(z) \sim \begin{cases} \sqrt{\kappa + \eta}, & \text{if } E \in [\lambda_-, \lambda_+], \\ \frac{\eta}{\sqrt{\kappa + \eta}}, & \text{if } E \notin [\lambda_-, \lambda_+]. \end{cases}$$

Similarly, we use the following two lemmas to collect the basic properties of $\mathcal{T}(z)$ defined in (4.21).

Lemma 4.9. For any $z \in \mathbf{D}(\tau)$ defined in (4.1), we have

$$|\mathcal{T}(z)| \sim 1, \quad |c^{1/2} - \mathcal{T}(z)| \sim \sqrt{\kappa + \eta},$$

as well as

$$\operatorname{Im} \mathcal{T}(z) \sim \begin{cases} \sqrt{\kappa + \eta}, & E \in [\lambda_-, \lambda_+]; \\ \frac{\eta}{\sqrt{\kappa + \eta}}, & E \notin [\lambda_-, \lambda_+]. \end{cases}$$

Similarly, we have

$$|\operatorname{Re} \mathcal{T}(z) - c^{1/2}| \sim \begin{cases} \frac{\eta}{\sqrt{\kappa + \eta}} + \kappa, & E \in [\lambda_-, \lambda_+], \\ \sqrt{\kappa + \eta}, & E \notin [\lambda_-, \lambda_+]. \end{cases} \quad (4.23)$$

Proof. By (4.21), it is easy to check that

$$\mathcal{T}(z) - c^{1/2} = \frac{z - \lambda_+ - i\sqrt{(\lambda_+ - x)(x - \lambda_-)}}{2c^{-1}}, \quad (4.24)$$

the rest of the proofs are based on the elementary calculations of (4.21) and (4.24). The details can be found in [5, Lemma 3.6]. \square

The next lemma provides the local results on the derivative of $\mathcal{T}(x)$ on the real axis.

Lemma 4.10. *For $d > c^{-1/4}$, denote $I_d := [x_-(d), x_+(d)]$, $x_\pm(d) := p(d) \pm N^{-1/2+\epsilon_0}(d - c^{-1/4})^{1/2}$, where ϵ_0 is defined in (3.1). Then $\forall x \in I_d$, we have that*

$$\mathcal{T}'(x) \sim (d - c^{-1/4})^{-1}.$$

Proof. By (3) of Lemma 4.6 and using the fact that $\mathcal{T}(p(d)) = d^{-2}$, then for $d > c^{-1/4}$,

$$\mathcal{T}(y) = d^{-2} \iff y = p(d). \quad (4.25)$$

By an elementary computation on (4.19), we have

$$\mathcal{T}'(p(d)) = \frac{1}{c^{-1} - d^4} \sim (d - c^{-1/4})^{-1}.$$

It is easy to check that there exists a constant $C > 0$, such that $|\mathcal{T}''(\xi)| \leq C$ for $\xi \in I_d$. Hence, we can conclude our proof using mean value theorem. \square

The perturbation identities play the key roles in our proofs, as it naturally provides us a way to incorporate the Green functions. We first derive the identities on which our analysis of the singular values and singular vectors rely. The following lemma uniquely characterizes the eigenvalues of \tilde{H} defined in (1.7), its proof can be found in [16, Lemma 6.1]. Recall (1.7), we have

Lemma 4.11. *Assume $\mu \in \mathbb{R}/\sigma(H)$ and $\det \mathbf{D} \neq 0$, then $\mu \in \sigma(\tilde{H})$ if and only if*

$$\det(\mathbf{U}^* G(\mu) \mathbf{U} + \mathbf{D}^{-1}) = 0. \quad (4.26)$$

The following lemma establishes the connection between the Green functions of H and \tilde{H} defined in (1.7).

Lemma 4.12. *For $z \in \mathbb{C}^+$, we have*

$$\tilde{G}(z) = G(z) - G(z) \mathbf{U} (\mathbf{D}^{-1} + \mathbf{U}^* G(z) \mathbf{U})^{-1} \mathbf{U}^* G(z) \quad (4.27)$$

and

$$\mathbf{U}^* \tilde{G}(z) \mathbf{U} = \mathbf{D}^{-1} - \mathbf{D}^{-1} (\mathbf{D}^{-1} + \mathbf{U}^* G(z) \mathbf{U})^{-1} \mathbf{D}^{-1}. \quad (4.28)$$

Proof. To prove (4.27), we write

$$\tilde{G}(z) = (H + \mathbf{U} \mathbf{D} \mathbf{U}^* - z)^{-1}.$$

The proof follows from the following identity (see [5, (3.36)]),

$$(A + SBT)^{-1} = A^{-1} - A^{-1} S (B^{-1} + T A^{-1} S)^{-1} T A^{-1},$$

with $A = H - z$, $B = \mathbf{D}^{-1}$, $S = \mathbf{U}$, $T = \mathbf{U}^*$. For the proof of (4.28), from (4.27), we have

$$\mathbf{U}^* \tilde{G}(z) \mathbf{U} = \mathbf{U}^* G(z) \mathbf{U} - \mathbf{U}^* G(z) \mathbf{U} (\mathbf{D}^{-1} + \mathbf{U}^* G(z) \mathbf{U})^{-1} \mathbf{U}^* G(z) \mathbf{U},$$

the proof then follows from the following identity (see the proof of [5, Lemma 3.11])

$$A - A(A + B)^{-1} A = B - B(A + B)^{-1} B,$$

with $A = \mathbf{U}^* G(z) \mathbf{U}$, $B = \mathbf{D}^{-1}$. \square

As we aim to investigate the local convergence of singular values and vectors, we will rely on the local analysis of the MP law. Our key tool is the anisotropic law, which was given by Knowles and Yin in [15] and later was used in a series of papers on covariance matrices [7, 8]. Our key ingredient is the following lemma, which can be found in [7, Lemma 2.1].

Lemma 4.13 (Anisotropic local law). *Recall (4.14), (4.16) and (4.17), for $\epsilon_1 > 0$ small enough, fix $\tau \gg \epsilon_1$, then for all $z \in \mathbf{D}(\tau)$ defined in (4.1), with $1 - N^{-D_1}$ probability, for any deterministic vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^{M+N}$, we have*

$$| \langle \mathbf{u}, \underline{\Sigma}^{-1}(G(z) - \Pi(z))\underline{\Sigma}^{-1}\mathbf{v} \rangle | \leq N^{\epsilon_1} \Psi(z), \quad (4.29)$$

$$|m_2(z) - m_{2c}(z)| \leq \frac{N^{\epsilon_1}}{N\eta}. \quad (4.30)$$

As discussed in [4, Section 1], the smallest scale η on which a deterministic limit of ESD is expected to emerge is when $\eta \gg N^{-1}$; below this scale the ESD fluctuates by the individual behavior of eigenvalues. Beyond the support of the limiting spectrum, we have stronger results all the way down to the real axis. More precisely, define the region

$$\tilde{\mathbf{D}}(\tau, \epsilon_1) := \{z \in \mathbb{C}^+ : \lambda_+ + N^{-2/3+\epsilon_1} \leq E \leq \tau, |z| \geq \tau, 0 < \eta \leq \tau^{-1}\}, \quad (4.31)$$

then we have the following stronger control on $\tilde{\mathbf{D}}(\tau)$, the proofs can be found in [4, Theorem 3.12].

Lemma 4.14. *For $z \in \tilde{\mathbf{D}}(\tau, \epsilon_1)$, with $1 - N^{-D_1}$ probability, we have*

$$| \langle u, \mathcal{G}_1(z)v \rangle - m_{1c}(z) \langle u, v \rangle | \leq N^{\epsilon_1} \sqrt{\frac{\text{Im } m_{1c}(z)}{N\eta}},$$

for all unit vectors $u, v \in \mathbb{R}^M$. Similarly, we have

$$| \langle u, \mathcal{G}_2(z)v \rangle - m_{2c}(z) \langle u, v \rangle | \leq N^{\epsilon_1} \sqrt{\frac{\text{Im } m_{2c}(z)}{N\eta}},$$

for all unit vectors $u, v \in \mathbb{R}^N$.

Next we will show that, the controls in Lemma 4.13 can be improved when $z \in \tilde{\mathbf{D}}(\tau, \epsilon_1)$. The proofs are very similar to that of Lemma 4.14, which can be found in [4, Section 6]. We only briefly sketch the proof and the details can be found in [4, Section 6].

Lemma 4.15 (Anisotropic law outside the spectrum). *For $z \in \tilde{\mathbf{D}}(\tau, \epsilon_1)$, with $1 - N^{-D_1}$ probability, for any deterministic vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^{M+N}$, we have*

$$| \langle \mathbf{u}, \underline{\Sigma}^{-1}(G(z) - \Pi(z))\underline{\Sigma}^{-1}\mathbf{v} \rangle | \leq N^{\epsilon_1} \sqrt{\frac{\text{Im } m_{2c}(z)}{N\eta}}. \quad (4.32)$$

Proof. When $\eta \geq \kappa \geq N^{-2/3+\epsilon_0}$, we have

$$\frac{\text{Im } m_{2c}(z)}{N\eta} \sim \frac{1}{N\sqrt{\kappa + \eta}} > \frac{1}{N^2\eta^{1/2}N^{-1}} > \frac{1}{N^2\eta^2},$$

where we use Lemma 4.8. Now we deal with the case $\eta < \kappa \leq \tau^{-1}$, again by Lemma 4.8, we have

$$\sqrt{\frac{\text{Im } m_{2c}(z)}{N\eta}} \sim N^{-1/2}\kappa^{-1/4}. \quad (4.33)$$

Denote $\eta_0 := N^{-1/2}\kappa^{1/4}$, by the definition of $\tilde{\mathbf{D}}(\tau, \epsilon_1)$, we have that $\eta_0 \leq \kappa$. It is easy to check that (4.33) holds true when $\eta \geq \eta_0$. Therefore, we only need to focus on the case when $\eta \leq \eta_0$. Denote the following two spectral parameters

$$z := E + i\eta, \quad z_0 := E + i\eta_0,$$

The rest of the proofs leave to compare $G(z)$ with $G(z_0)$, $m_{1,2c}(z)$ with $m_{1,2c}(z_0)$. The proof is similar to that of [4, Theorem 3.12], to be specific, (6.2) and (6.3). We omit the details and refer to the proofs in the Section 6 of [4]. \square

As we will use the above control when $z \in \mathbb{R}$, the following lemma summarizes the results when z is restricted to the real axis.

Lemma 4.16. *For $z > \lambda_+ + N^{-2/3+\epsilon_1}$, with $1 - N^{-D_1}$ probability, for any deterministic vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^{M+N}$, we have*

$$| \langle \mathbf{u}, \underline{\Sigma}^{-1}(G(z) - \Pi(z)) \underline{\Sigma}^{-1} \mathbf{v} \rangle | \leq N^{-1/2+\epsilon_1} \kappa^{-1/4}.$$

Similar results hold for Lemma 4.14 by replacing the bounds with $N^{-1/2+\epsilon_1} \kappa^{-1/4}$.

Proof. Denote the spectral $z_0 = E + i\eta$, where η can be sufficiently small, for example N^{-D_1} . The result follows from Lemma 4.15, the triangle inequality and the following two facts:

$$|m_{1c}(z) - m_{1c}(z_0)| \leq C\eta, \quad |m_{2c}(z) - m_{2c}(z_0)| \leq C\eta,$$

where $C > 0$ is some constant and

$$| \langle \mathbf{u}, G(z) - G(z_0) \mathbf{v} \rangle | \leq C\eta.$$

□

The consequent results of Lemma 4.13 and Lemma 4.14 are the rigidity of eigenvalues and the isotropic delocalization. They are proved in [4, Theorem 2.10 and 2.8] respectively. Denote the nontrivial classical eigenvalue locations $\gamma_1 \geq \gamma_2 \geq \dots \geq \gamma_K$ of X^*X as

$$\int_{\gamma_i}^{\infty} d\rho = \frac{i}{N},$$

where ρ is defined in (4.12). Then near the right edge λ_+ of XX^* , we have

Lemma 4.17 (Rigidity of eigenvalues). *For $i \leq C$, where $C > 0$ is a large constant, with $1 - N^{-D_1}$ probability, we have*

$$|\lambda_i - \gamma_i| \leq N^{-2/3+\epsilon_1} i^{-1/3}.$$

Denote $\xi_i \in \mathbb{R}^M, \zeta_i \in \mathbb{R}^N$ as the singular vectors of X , the following isotropic delocalization bounds implies that the entries $\xi_i(k), \zeta_i(k)$ of the singular vectors are strongly oscillating in the sense that $|\sum \xi_i(k)| \prec 1$ but $\sum |\xi_i(k)| \succ N^{1/2}$, which implies the completely delocalization of the singular vectors. Due to the purpose of our application, we only consider the singular vectors near the right edge.

Lemma 4.18 (Isotropic delocalization). *For $i \leq C$, where $C > 0$ is a large constant, for any normalized vector $m \in \mathbb{R}^M, n \in \mathbb{R}^N$, with $1 - N^{-D_1}$ probability, we have*

$$\max_i \{ | \langle \xi_i, m \rangle |^2 + | \langle \zeta_i, n \rangle |^2 \} \leq N^{-1+\epsilon_1}.$$

5 Singular values: proof of Theorem 3.4

In this section, we focus on the singular values of \tilde{S} and prove Theorem 3.4. We will follow the basic idea of [16]. However, as we use the linearization matrix (1.7), we need to modify many of the proofs. This is due to the fact that many of the properties on which their analysis rely do not hold true here. For example, our matrix \mathbf{D} is not diagonal, therefore, many of the expressions will become matrix forms instead of scalars. In particular, to analyze (4.26), they only need to deal with the diagonal elements but we need to control the whole matrix. A second deviation from their proof is that, in order to locate the eigenvalues of $\tilde{S}\tilde{S}^*$, we need to construct some permissible regions in which the eigenvalues are allowed to lie. The construction of such regions are different from [16], which makes our proof slightly easier due to our application purpose.

We will make use of the following interlacing theorem for rectangular matrices, which is purely deterministic. It naturally provides some controls of the singular values of \tilde{S} . The proofs are elementary by using Courant-Fischer lemma, for example, see [23, Exercise 1.3.22].

Lemma 5.1 (Weyl interlacing theorem). *For any $M \times N$ matrices A, B , denote $\sigma_i(A)$ as the i -th largest singular value of A , then we have*

$$\sigma_{i+j-1}(A+B) \leq \sigma_i(A) + \sigma_j(B), \quad 1 \leq i, j, i+j-1 \leq K.$$

To illustrate our idea, we firstly deal with the case when $r = 1, U = u, V = v$ in (1.3). Then

$$\tilde{S} = X + duv^*, \quad \mu := \mu_1(\tilde{S}\tilde{S}^*). \quad (5.1)$$

We will use (4.26) as our key representation of the singular values of \tilde{S} . The advantage of using such an equation is that the singular values are determined by a deterministic equation.

Theorem 5.2. *Consider the model (5.1), then there exist ϵ_0 and large constant $C > 1$, such that $C\epsilon_1 < \epsilon_0$, when N is large enough, with $1 - N^{-D_1}$ probability, for $d \geq c^{-1/4} + N^{-1/3+\epsilon_0}$, we have*

$$|\mu - p(d)| \leq (d - c^{-1/4})^{1/2} N^{-1/2+C\epsilon_0}, \quad (5.2)$$

where $p(d)$ is defined in (1.11). Furthermore, when $d < c^{-1/4} + N^{-1/3+\epsilon_0}$, we have

$$|\mu - \lambda_+| \leq N^{-2/3+C\epsilon_0}, \quad (5.3)$$

where λ_+ is defined in (1.5).

Proof. By (4.14) and (4.26), recall (1.7), under the assumption that $\mu \notin \sigma(H)$, μ is an eigenvalue of \tilde{H} , if and only if

$$(u^* \mathcal{G}_1(\mu)u)(v^* \mathcal{G}_2(\mu)v) - (u^* E_1(\mu)v + \mu^{-1/2}d^{-1})(v^* E_2(\mu)u + \mu^{-1/2}d^{-1}) = 0, \quad (5.4)$$

where $E_1(\mu)$, $E_2(\mu)$ are defined as

$$E_1(\mu) := \mu^{-1/2} \mathcal{G}_1(\mu)X, \quad E_2(\mu) := \mu^{-1/2} X^* \mathcal{G}_1(\mu).$$

Denote

$$M(\mu) := T_1(\mu) + R(\mu),$$

where $T_1(\mu)$, $R(\mu)$ are defined as

$$T_1(\mu) := \mu(u^* \mathcal{G}_1(\mu)u)(v^* \mathcal{G}_2(\mu)v),$$

$$R(\mu) := -\mu^{1/2}(d^{-1}u^* E_1(\mu)v + d^{-1}v^* E_2(\mu)u + \mu^{1/2}(u^* E_1(\mu)v)(v^* E_2(\mu)u)).$$

Therefore, by (5.4), μ is determined by the following equation

$$M(\mu) = \mathcal{T}(p), \quad p \equiv p(d), \quad (5.5)$$

where $\mathcal{T}(p)$ is defined in (4.21) and $\mathcal{T}(p) = d^{-2}$. Now we choose $\epsilon_2 \geq C\epsilon_1$ and denote

$$I_p := [x_-(p), x_+(p)], \quad x_{\pm}(p) := p(d) \pm N^{-1/2+\epsilon_2}(d - c^{-1/4})^{1/2}. \quad (5.6)$$

When $d \geq c^{-1/4} + N^{-1/3+\epsilon_0}$, using the fact that $p(d)$ is strictly increasing and $p(c^{-1/4}) = \lambda_+$, we have $p(d) > \lambda_+ + N^{-2/3+\epsilon_1}$. Next, we claim that:

$$M(x_+(p)) < \mathcal{T}(p) < M(x_-(p)). \quad (5.7)$$

We first show that (5.7) implies (5.2). Assume $\epsilon_0 \gg \epsilon_2$, we have $x_-(p) > \lambda_+ + N^{-2/3+\epsilon_1}$. Using Lemma 5.1 and Lemma 4.17, we have $\mu_2(\tilde{S}\tilde{S}^*) \leq \lambda_1(XX^*) \leq \lambda_+ + N^{-2/3+\epsilon_1}$ holds with $1 - N^{-D_1}$ probability. Therefore, there exists one and only one eigenvalue on $(\lambda_+ + N^{-2/3+\epsilon_1}, \infty)$. (5.7) shows that this eigenvalue lies in I_p , which concludes the proof of (5.2). We now prove the claim (5.7). By definition, we have

$$\mathcal{T}(p) - M(x_+(p)) = \mathcal{T}(x_+(p)) - \mathcal{T}'(\xi)N^{-1/2+\epsilon_2}(d - c^{-1/4})^{1/2} - M(x_+(p)), \quad (5.8)$$

where $\xi \in (p, x_+(p))$. By Lemma 4.16, when $\mu > \lambda_+$, with $1 - N^{-D_1}$ probability, we have

$$|u^* E_1(\mu)v| \leq N^{-1/2+\epsilon_1} \kappa^{-1/4}, \quad |v^* E_2(\mu)u| \leq N^{-1/2+\epsilon_1} \kappa^{-1/4}.$$

Therefore, with $1 - N^{-D_1}$ probability, we have

$$R(\mu) = O(N^{-1/2+\epsilon_1}\kappa^{-1/4}). \quad (5.9)$$

Similarly, using Lemma 4.8, Lemma 4.16 and triangle inequality, with $1 - N^{-D_1}$ probability, we have

$$T_1(\mu) - \mathcal{T}(\mu) = O(N^{-1/2+\epsilon_1}\kappa^{-1/4}). \quad (5.10)$$

By (5.8), (5.9) and (5.10), with $1 - N^{-D_1}$ probability, we have

$$\mathcal{T}(p) - M(x_+(p)) = -\mathcal{T}'(\xi)N^{-1/2+\epsilon_2}(d - c^{-1/4})^{1/2} + O(2N^{-1/2+\epsilon_1}\kappa_{x_+}^{-1/4}),$$

where we have

$$\kappa_{x_+} = p(d) - p(c^{-1/4}) + N^{-1/2+\epsilon_2}(d - c^{-1/4})^{1/2} \sim (d - c^{-1/4})^2 + N^{-1/2+\epsilon_2}(d - c^{-1/4})^{1/2},$$

where we use the mean value theorem. By Lemma 4.10, we have

$$\mathcal{T}'(\xi) \sim (d - c^{-1/4})^{-1}, \quad (5.11)$$

then by (3) of Lemma 4.6, we have

$$\mathcal{T}(p) - M(x_+(p)) \geq N^{-1/2+\epsilon_2}(d - c^{-1/4})^{-1/2} - N^{-1/2+\epsilon_1}(d - c^{-1/4})^{-1/2} > 0,$$

where we use the assumption $\epsilon_2 > C\epsilon_1$ for some large constant $C > 1$. Similarly, we can show the other direction. As we assume $\epsilon_0 > \epsilon_2$, this concludes the proof of (5.7).

For the proof of (5.3), denote

$$d_0 = c^{-1/4} + N^{-1/3+\epsilon_0},$$

and its associated eigenvalue by μ_0 . Choose some fixed constant $C_1 > 2$, when $d_0 - d \leq N^{-1/3+C_1\epsilon_0}$, there exists some constant $C > 0$, by (5.5), we have

$$M(\mu) - M(\mu_0) = d^{-2} - d_0^{-2} \leq CN^{-1/3+C_1\epsilon_0}.$$

Then using (5.9) and (5.10)¹, by (5.5) we find that

$$\mathcal{T}(\mu) - \mathcal{T}(\mu_0) \leq CN^{-1/3+C_1\epsilon_0} + O(N^{-1/2+\epsilon_1}\kappa_\mu^{-1/4}) + O(N^{-1/2+\epsilon_1}\kappa_{\mu_0}^{-1/4}). \quad (5.12)$$

Using (5.2) for μ_0 , we find that for some constant $C_2 > 0$,

$$N^{-1/2+\epsilon_1}\kappa_{\mu_0}^{-1/4} \leq N^{-1/3+C_2\epsilon_0}. \quad (5.13)$$

(5.3) holds immediately when $\kappa_\mu \leq N^{-2/3+C_2\epsilon_0}$. When $\kappa_\mu \geq N^{-2/3+C_2\epsilon_0}$, by (5.12) and (5.13), we have

$$\mathcal{T}'(\xi)(\mu - \mu_0) \leq N^{-1/3+C_2\epsilon_0}, \quad \xi \text{ is between } \mu \text{ and } \mu_0.$$

By (5.11), this yields that

$$|\mu - \mu_0| \leq N^{-2/3+C_2\epsilon_0},$$

which implies (5.3). What remains is the case $d_0 - d > N^{-1/3+C_1\epsilon_0}$, we first observe that

$$0 < \frac{1}{d^2} - \frac{1}{d_0^2} = M(\mu) - M(\mu_0) = \mathcal{T}(\mu) - \mathcal{T}(\mu_0) + O(N^{-1/3+\epsilon_1-C_1\epsilon_0/4}),$$

which implies that

$$\mathcal{T}(\mu) - \mathcal{T}(\mu_0) \geq 0.$$

Hence, we conclude that $\mu \leq \mu_0$ using the fact that $\mu_0 > \lambda_+$ with $1 - N^{-D_1}$ probability and (3) of Lemma 4.6. Denote λ_1 as the largest eigenvalue of XX^* , we have

$$\mu - \lambda_1 \leq \mu_0 - p(d_0) + p(d_0) - p(c^{-1/4}) + p(c^{-1/4}) - \lambda_1,$$

¹If μ is already in the bulk, the proof is done. So we only need to discuss the case when $\mu \in \tilde{\mathbf{D}}(\tau, \epsilon_1)$ defined in (4.31).

It is easy to check that

$$p'(x) \sim (x - c^{-1/4}). \quad (5.14)$$

Therefore, combine with Lemma 4.17 and (5.2), with $1 - N^{-D_1}$ probability, we have

$$\mu - \lambda_1 \leq N^{-2/3+C\epsilon_0}.$$

For the other direction, we actually have $\mu \geq \lambda_1$ by the following fact: for any two $M \times N$ rectangle matrices A, B , we have

$$\sigma_1(A + B) \geq \sigma_1(A) + \sigma_K(B).$$

By Lemma 4.17, we conclude the proof of (5.3). \square

Next we will extend the proof to the rank r case using the basic strategy for the rank one case, especially (5.5) and (5.6). The main idea is to use a standard counting argument, where we follow the idea of [16, Section 6] but slightly modify their proofs. It relies on two main steps: (i) fix a configuration independent of N , establishing two permissible regions, $\Gamma(\mathbf{d})$ of k^+ components and I_0 , where the outliers of $\tilde{S}\tilde{S}^*$ are allowed to lie in $\Gamma(\mathbf{d})$ and each component contains precisely one eigenvalue and the $r - k^+$ non-outliers lie in I_0 ; (ii) a continuity argument where the result of (i) can be extended to arbitrary N -dependent \mathbf{D} .

As we have seen from the proofs of (5.2) and (5.3), the following $2r \times 2r$ matrix plays the key role in our analysis

$$M^r(x) := \mathbf{U}^* G(x) \mathbf{U} + \mathbf{D}^{-1}. \quad (5.15)$$

By Lemma 4.11, $x \in \sigma(\tilde{S}\tilde{S}^*)$ if and only if $\det M^r(x) = 0$. Using the anisotropic law Lemma 4.15, we find that $x^r T^r(x) \approx \det M^r(x)$, where $T^r(x)$ is defined in (4.18). As $T^r(x)$ behaves differently in $\Gamma(\mathbf{d})$ and I_0 , we will use different strategies to prove (3.6) and (3.7).

We remark that, our discussion is slightly easier than [16, Section 6], in particular the continuity argument of the non-outliers. The reason is, for the application purpose, we only need the result of (3.7) to locate the eigenvalues around λ_+ . However, in [16], they have stronger results to stick the eigenvalues of $\tilde{S}\tilde{S}^*$ around those of XX^* . We will not pursue this generalization in this paper.

Proof of Theorem 3.4. Recall that $k^+ = |\mathcal{O}|$, where \mathcal{O} is defined in (3.1). Define $k^0 := r - k^+$ and write

$$\mathbf{d} = (d_1, \dots, d_r) = (\mathbf{d}^0, \mathbf{d}^+), \quad \mathbf{d}^\sigma = (d_1^\sigma, \dots, d_{k^\sigma}^\sigma), \quad \sigma = 0, +,$$

where we adapt the convention

$$d_{k^0}^0 \leq \dots \leq d_1^0 \leq c^{1/4} < d_{k^+}^+ \leq \dots \leq d_1^+, \quad k^0 + k^+ = r.$$

Recall that $p(d)$ attains its minimum at $d = c^{-1/4}$, we denote that for any d_i^0, d_j^+ , we call them a **couple** if

$$d_i^0 d_j^+ = \frac{1}{\sqrt{c}}, \quad 1 \leq i \leq k^0, \quad 1 \leq j \leq k^+.$$

By elementary calculation, we find that for each couple,

$$p(d_i^0) = p(d_j^+).$$

We denote

$$L_i^+ := \{p(d_j^+), j = 1, 2, \dots, n_i\}, \quad \text{where } d_j^+, j = 1, \dots, n_i \text{ are in a couple with } d_i^+.$$

Next we define the sets

$$\mathcal{D}^+(\epsilon_0) := \{\mathbf{d}^+ : c^{-1/4} + N^{-1/3+\epsilon_0} \leq d_i^+ \leq \tau^{-1} - 1, \quad i = 1, \dots, k^+\}, \quad (5.16)$$

$$\mathcal{D}^0(\epsilon_0) := \{\mathbf{d}^0 : d_i^0 < c^{-1/4} + N^{-1/3+\epsilon_0}, \quad i = 1, \dots, k_0\}, \quad (5.17)$$

and the sets of allowed \mathbf{d}' s

$$\mathcal{D}(\epsilon_0) := \{(\mathbf{d}^0, \mathbf{d}^+) : \mathbf{d}^\sigma \in \mathcal{D}^\sigma(\epsilon_0), \quad \sigma = +, 0\}.$$

Similar to (5.6), we denote the following sequence of intervals

$$m_i^+(\mathbf{d}) := [p(d_i^+) - N^{-1/2+\epsilon_3}(d_i^+ - c^{-1/4})^{1/2}, p(d_i^+) + N^{-1/2+\epsilon_3}(d_i^+ - c^{-1/4})^{1/2}], \quad (5.18)$$

where ϵ_3 satisfies the following condition

$$C\epsilon_1 < \epsilon_3 < \frac{1}{2}\epsilon_0, \quad C > 2 \text{ is some large constant.} \quad (5.19)$$

Furthermore, we will say d_i^+, d_j^0 are in a **class** if $p(d_j^0) \in m_i^+(\mathbf{d})$. Denote

$$C_i^+ := \{p(d_j^0), j = 1, 2, \dots, n_i\}, \text{ where } d_j^0, j = 1, \dots, n_i \text{ are in the same class with } d_i^+.$$

To remove the influence of the class, we now define the sets

$$I_i^+(\mathbf{d}) := [p(d_i^+) - N^{-1/2+\epsilon_3}(d_i^+ - c^{-1/4})^{1/2}, p(d_i^+) + N^{-1/2+\epsilon_3}(d_i^+ - c^{-1/4})^{1/2}] \cap (C_i^+)^c \cup (L_i^+). \quad (5.20)$$

For $\mathbf{d} \in \mathcal{D}(\epsilon_0)$, we denote

$$\Gamma(\mathbf{d}) := \cup_{i=1}^{k^+} I_i^+(\mathbf{d}).$$

and

$$I^0 := [\lambda_+ - N^{-2/3+C'\epsilon_0}, \lambda_+ + N^{-2/3+C'\epsilon_0}],$$

where C' satisfying

$$C' > \max\{2, \frac{4\epsilon_1}{\epsilon_0}\}. \quad (5.21)$$

For a first step, we show that $\Gamma(\mathbf{d})$ is our permissible region which keeps track of the outlier eigenvalues of $\tilde{S}\tilde{S}^*$. And the rest of the eigenvalues corresponding to $\mathcal{D}^0(\epsilon_0)$ will lie in I^0 . We fix a configuration $\mathbf{d}(0) \equiv \mathbf{d}$ that is independent of N in this step.

Lemma 5.3. *For any $\mathbf{d} \in \mathcal{D}(\epsilon_0)$, with $1 - N^{-D_1}$ probability, we have*

$$\sigma^+(\tilde{S}\tilde{S}^*) \subset \Gamma(\mathbf{d}), \quad (5.22)$$

where $\sigma^+(\tilde{S}\tilde{S}^*)$ is the set of the outlier eigenvalues of $\tilde{S}\tilde{S}^*$. Moreover, each interval $I_i^+(\mathbf{d})$ contains precisely one eigenvalue of $\tilde{S}\tilde{S}^*$, $i = 1, 2, \dots, k^+$. Furthermore, we have

$$\sigma^o(\tilde{S}\tilde{S}^*) \subset I^0, \quad (5.23)$$

where $\sigma^o(\tilde{S}\tilde{S}^*)$ is the set of the non-outlier eigenvalues corresponding to $\mathcal{D}^0(\epsilon_0)$.

Proof. Denote

$$S_b := p(d_{k^+}^+) - N^{-1/2+\epsilon_3}(d_{k^+}^+ - c^{-1/4})^{1/2}.$$

In order to prove (5.22), we first consider the case when $x > S_b$. It is notable that $x \notin \sigma(XX^*)$ by Lemma 4.17 and (5.19). Recall (4.17), using the fact r is bounded and Lemma 4.16, with $1 - N^{-D_1}$ probability, we have

$$M^r(x) = \mathbf{U}^* \Pi(x) \mathbf{U} + \mathbf{D}^{-1} + O(N^{-1/2+\epsilon_1} \kappa^{-1/4}). \quad (5.24)$$

It is well-known that if $\lambda \in \sigma(A + B)$ then $\text{dist}(\lambda, \sigma(A)) \leq \|B\|$; therefore, we have that $\mu_i(\tilde{S}\tilde{S}^*) \leq \tau^{-1}$, $i = 1, \dots, K$ for $\tau > 0$ defined in (4.1). By (5.19) and a similar discussion as (5.11), with $1 - N^{-D_1}$ probability, we have

$$|T^r(x)| \geq N^{-1/2+(C-1)\epsilon_1} \kappa^{-1/4}, \text{ if } x \in [S_b, \tau^{-1}]/\Gamma(\mathbf{d}). \quad (5.25)$$

where $T^r(x)$ is defined in (4.18). Using the formula

$$\det \begin{bmatrix} xI_r & \text{diag}(\alpha_1, \dots, \alpha_r) \\ \text{diag}(\alpha_1, \dots, \alpha_r) & yI_r \end{bmatrix} = \prod_{i=1}^r (xy - \alpha_i^2),$$

we conclude that

$$\det(\mathbf{D}^{-1} + \mathbf{U}^* \Pi(x) \mathbf{U}) = \prod_{k=1}^r (m_{1c}(x) m_{2c}(x) - x^{-1} d_k^{-2}) = x^r T^r(x). \quad (5.26)$$

Therefore, by (5.24), (5.25), (5.26) and (1) of Lemma 4.6, we conclude that $M^r(x)$ is nonsingular when $x \in [S_b, \tau^{-1}]/\Gamma(\mathbf{d})$. This concludes the proof of (5.22) by Lemma 4.11. Next we will use Roché's theorem to show that inside the permissible region, each interval $I_i^+(\mathbf{d})$ contains precisely one eigenvalue of $\tilde{S}\tilde{S}^*$. Let $i \in \{1, \dots, k^+\}$ and pick a small N -independent counterclockwise (positive oriented) contour $\mathcal{C} \subset \mathbb{C}/[(1 - c^{-1/2})^2, (1 + c^{-1/2})^2]$ that encloses $p(d_i^+)$ but no other point. For large enough N , define

$$f(z) := \det(M^r(z)), \quad g(z) := \det(T^r(z)).$$

By the definition of determinant, the functions g, f are holomorphic on and inside \mathcal{C} . And $g(z)$ has precisely one zero $z = p(d_i^+)$ inside \mathcal{C} . On \mathcal{C} , it is easy to check that

$$\min_{z \in \mathcal{C}} |g(z)| \geq c > 0, \quad |g(z) - f(z)| \leq N^{-1/2 + \epsilon_1} \kappa^{-1/4},$$

where we use (5.24). Hence, $f(z)$ has only one eigenvalue in $I_i^+(\mathbf{d})$ according to Rouché's theorem. In order to prove (5.23), using the following fact: for any two $M \times N$ rectangular matrices A, B , we have

$$\sigma_i(A + B) \geq \sigma_i(A) + \sigma_K(B), \quad i = 1, \dots, K,$$

and Lemma 4.17, we find that

$$\mu_i \geq \lambda_+ - N^{-2/3 + C'\epsilon_0}, \quad i = k^+ + 1, \dots, r. \quad (5.27)$$

As we now prove the non-outliers, we assume that $S_b > \lambda_+ + N^{-2/3 + C'\epsilon_0}$, otherwise the proof is already done. Now we assume $x \notin I_0$, by (5.27), we only need to discuss when $x \in (\lambda_+ + N^{-2/3 + C'\epsilon_0}, S_b)$. In this case, we will prove that $M^r(x)$ is non-singular by comparing with $M^r(z)$, where $z = x + iN^{-2/3 - \epsilon_4}$, where $\epsilon_4 < \epsilon_1$ is some small positive constant. Denote the spectral decomposition of $G(z)$ as

$$G(z) = \sum_k \frac{1}{\lambda_k - z} \mathbf{g}_\alpha \mathbf{g}_\alpha^*, \quad \mathbf{g}_\alpha \in \mathbb{R}^{M+N}.$$

Denote $\mathbf{u}_i, i = 1, \dots, 2r$ as the i -th column in \mathbf{U} defined in (1.8) and abbreviate $\mathbf{u}_i^* G(z) \mathbf{u}_j$ as $G_{\mathbf{u}_i \mathbf{u}_j}(z)$, and $\eta := N^{-2/3 - \epsilon_4}$, then we have

$$\begin{aligned} |G_{\mathbf{u}_i \mathbf{u}_j}(x) - G_{\mathbf{u}_i \mathbf{u}_j}(x + i\eta)| &\leq \sum_\alpha \frac{|\langle \mathbf{g}_\alpha, \mathbf{u}_i \rangle|^2 + |\langle \mathbf{g}_\alpha, \mathbf{u}_j \rangle|^2}{2} \left| \frac{1}{\lambda_\alpha - x} - \frac{1}{\lambda_\alpha - x - i\eta} \right| \\ &\leq \sum_\alpha (|\langle \mathbf{g}_\alpha, \mathbf{u}_i \rangle|^2 + |\langle \mathbf{g}_\alpha, \mathbf{u}_j \rangle|^2)^2 \frac{\eta}{(\lambda_\alpha - x)^2 + \eta^2} \\ &= \text{Im } G_{\mathbf{u}_i \mathbf{u}_i}(x + i\eta) + \text{Im } G_{\mathbf{u}_j \mathbf{u}_j}(x + i\eta), \end{aligned}$$

where in the second inequality we use the fact $x > \lambda_+ + N^{-2/3 + C'\epsilon_0}$. Therefore, by Lemma 4.15, we have

$$M^r(x) = M^r(z) + O(N^{\epsilon_1} \left(\text{Im } m_{2c}(z) + \sqrt{\frac{\text{Im } m_{2c}(z)}{N\eta}} \right)).$$

Using Lemma 4.8 and a similar discussion of (5.24), we have

$$M^r(x) = T^r(z) + O(N^{-1/3} (N^{-C'\epsilon_0/4} + N^{\epsilon_1 - C'\epsilon_0/4})).$$

By Lemma 4.6 and Lemma 4.9, we find that

$$|T^r(z)| \sim (x - c^{-1/4}) \geq N^{-1/3 + \frac{C'\epsilon_0}{2}},$$

where we use the assumption that $x > \lambda_+ + N^{-2/3 + C'\epsilon_0}$. Therefore, $M^r(x)$ is nonsingular as we have assumed (5.21). This concludes the proof of (5.23). \square

In the second step we will extend the proofs to any configuration $\mathbf{d}(1)$ depending on N by using the continuity argument. This is done by a bootstrap argument by choosing a continuous path. We first deal with (3.6). As r is finite, we can choose a path $(\mathbf{d}(t) : 0 \leq t \leq 1)$ connecting $\mathbf{d}(0)$ and $\mathbf{d}(1)$ having the following properties:

- (i) For all $t \in [0, 1]$, the point $\mathbf{d}(t) \in \mathcal{D}(\epsilon_0)$.
- (ii) If $I_i^+(\mathbf{d}(1)) \cap I_j^+(\mathbf{d}(1)) = \emptyset$ for a pair $1 \leq i < j \leq k^+$, then $I_i^+(\mathbf{d}(t)) \cap I_j^+(\mathbf{d}(t)) = \emptyset$ for all $t \in [0, 1]$.

Denote $\tilde{S}(t) := X + UD(t)V$, where $D(t)$ is a diagonal matrix with elements $d_1(t), \dots, d_r(t)$. As the mapping $t \rightarrow \tilde{S}(t)$ is continuous, we find that $\mu_i(t)$ is continuous in $t \in [0, 1]$ for all i , where $\mu_i(t)$ are the eigenvalues of $\tilde{S}(t)\tilde{S}^*(t)$. Moreover, by Lemma 5.3, we have

$$\sigma^+(\tilde{S}(t)\tilde{S}^*(t)) \subset \Gamma(\mathbf{d}(t)), \forall t \in [0, 1]. \quad (5.28)$$

In the case when the k^+ intervals are disjoint (i.e. they satisfy the non-overlapping condition (1.16)), we have

$$\mu_i(t) \in I_i^+(\mathbf{d}(t)), \quad d \in [0, 1],$$

where we use property (ii) of the continuous path, (5.28) and the continuity of $\mu_i(t)$. In particular, it holds true for $\mathbf{d}(1)$. Now we consider the case when they are not disjoint. Define \mathcal{B} as a partition of $\{1, \dots, k^+\}$ and denote the equivalent relation as

$$i \equiv j \quad \text{if} \quad I_i^+(\mathbf{d}(1)) \cap I_j^+(\mathbf{d}(1)) \neq \emptyset.$$

Therefore, we can decompose $\mathcal{B} = \cup_i \mathcal{B}_i$. It is notable that each \mathcal{B}_i contains a sequence of consecutive integers. Choose any $j \in \mathcal{B}_i$, without loss of generality, we assume j is not the smallest element in \mathcal{B}_i . Since they are not disjoint, we have

$$\frac{1}{p'(d_{j-1}^+)}(d_{j-1}^+ - d_j^+) \leq p(d_{j-1}^+) - p(d_j^+) \leq 2N^{-1/2+\epsilon_1+\epsilon_3}(d_{j-1}^+ - c^{-1/4})^{1/2},$$

where we use the fact that $p(x)$ is monotone increasing when $x > c^{-1/4}$ and (5.20). This implies that

$$d_{j-1}^+ - d_j^+ \leq CN^{-1/2+\epsilon_1+\epsilon_3}(d_j^+ - c^{-1/4})^{-1/2},$$

for some constant $C > 0$. By (5.19), we have

$$(d_{j-1}^+ - c^{-1/4})^{1/2} \leq (d_j^+ - c^{-1/4})^{1/2} \left(1 + \frac{d_{j-1}^+ - d_j^+}{d_j^+ - c^{-1/4}}\right) \leq (d_j^+ - c^{-1/4})^{1/2}(1 + o(1)).$$

Therefore, by repeating the process for the remaining $j \in \mathcal{B}_i$, we find

$$\text{diam}(\cup_{j \in \mathcal{B}_i} I_j^+(\mathbf{d}(1))) \leq CN^{-1/2+C\epsilon_0} \min_{j \in \mathcal{B}_i} (d_j^+(1) - c^{-1/4})^{1/2}(1 + o(1)),$$

where we use the fact that $r = O(1)$. This immediately yields that

$$|\mu_i(1) - p(d_i^+)| \leq N^{-1/2+C\epsilon_0}(d_i^+(1) - 1)^{1/2},$$

for some constant $C > 0$. This completes the proof of (3.6). Finally, we will deal with the extremal bulk eigenvalues (3.7). By the continuity of $\mu_i(t)$ and Lemma 5.3, we have

$$\sigma^0(\tilde{S}(t)\tilde{S}^*(t)) \subset I^0(t), \quad t \in [0, 1]. \quad (5.29)$$

In particular it holds true for $\mathbf{d}(1)$. This concludes our proof. \square

6 Singular vectors: proof of Theorem 3.5

In this section, we focus on the local behavior of singular vectors. We will follow the discussion of [5, Section 5 and 6]. We first deal with the outlier singular vectors and then the non-outlier ones. For the outlier singular vectors, under the assumption of (3.4), we will use an integral representation (6.6) with some well-chosen contour. Using the residual theorem and a resolvent expansion, we can conclude our proof. However, our representation is much more complicated than the discussion in [5], for example, our decomposition of the integral expression (6.9) contains more complicated terms due to the fact that \mathbf{D} is not a diagonal matrix. Hence, we need to explicitly write down the entries of the inverse of the matrix. For the non-outlier singular vectors, the residual theorem will not work as there is no pole inside the contour. Instead, we will delocalize the singular vectors using the spectral decomposition.

6.1 Outlier singular vectors

In this section, we will deal with the outlier singular vectors and prove (3.8) and (3.9). Due to similarity, we only prove (3.9) and point out the differences from (3.8).

Proof of (3.9). It is notable that, by Lemma 4.15 and Theorem 3.4, for $i \in \mathcal{O}$, there exists a constant $C > 0$, for N large enough, with $1 - N^{-D_1}$ probability, we can choose an event Ξ satisfying the following conditions:

(i) For all $z \in \tilde{\mathbf{D}}(\tau, \epsilon_1)$ defined in (4.31)

$$\mathbf{1}(\Xi) |(V^* \mathcal{G}_2(z) V)_{ij} - m_{2c}(z) \delta_{ij}| \leq (\kappa + \eta)^{-1/4} N^{-1/2+C\epsilon_1}. \quad (6.1)$$

(ii) For all $i \in \mathcal{O}$, we have

$$\mathbf{1}(\Xi) |\mu_i - p(d_i)| \leq (d_i - c^{-1/4})^{1/2} N^{-1/2+C\epsilon_0}.$$

(iii) Recall that $k^+ = |\mathcal{O}|$, we have

$$\mathbf{1}(\Xi) |\mu_\alpha - \lambda_+| \leq N^{-2/3+C\epsilon_0}, \quad \alpha = k^+ + 1, \dots, r.$$

Next we will restrict our discussion on the event Ξ . Recall (3.5) and for $A \subset \mathcal{O}$, we define for each $i \in A$ the radius

$$\rho_i := \frac{\nu_i \wedge (d_i - c^{-1/4})}{2}. \quad (6.2)$$

By [5, (5.10)], we have

$$\rho_i \geq \frac{1}{2} (d_i - c^{-1/4})^{1/2} N^{-1/2+\epsilon_0}. \quad (6.3)$$

We define the contour

$$\Gamma := \partial \Upsilon, \quad (6.4)$$

as the boundary of the union of discs $\Upsilon := \cup_{i \in A} B_{\rho_i}(d_i)$, where $B_\rho(d)$ is the open disc of radius ρ around d . We summarize the basic properties of Υ as the following lemma, its proof can be found in [5, Lemma 5.4 and 5.5].

Lemma 6.1. (i). Recall (1.11) and (4.31), we have

$$\overline{p(\Upsilon)} \subset \tilde{\mathbf{D}}(\tau, \epsilon_1).$$

(ii). Each outlier $\{\mu_i\}_{i \in A}$ lies in $p(\Upsilon)$, and all the other eigenvalues of $\tilde{S}\tilde{S}^*$ lie in the complement of $\overline{p(\Upsilon)}$.

Armed with the above results, we now start the proof of the outlier singular vectors. Our starting point is an integral representation of the singular vectors. By definition, we have

$$v_i^* \tilde{\mathcal{G}}_2 v_j = \mathbf{v}_i^* \tilde{G} \mathbf{v}_j, \quad (6.5)$$

where $\mathbf{v}_i \in \mathbb{R}^{M+N}$ is the natural embedding of v_i by $\mathbf{v}_i = (0, v_i)^*$. Recall (3.3), by (4.15) and Cauchy's integral formula, we have

$$\mathbf{P}_r = -\frac{1}{2\pi i} \int_{p(\Gamma)} \tilde{G}_2(z) dz = -\frac{1}{2\pi i} \int_{\Gamma} \tilde{G}_2(p(\zeta)) p'(\zeta) d\zeta, \quad (6.6)$$

where Γ is defined in (6.4). By Lemma 4.12, Cauchy's integral formula and (6.6), we have

$$\langle v_i, \mathbf{P}_r v_j \rangle = \frac{1}{2\pi i} \int_{p(\Gamma)} [\mathbf{D}^{-1} (\mathbf{D}^{-1} + \mathbf{U}^* G(z) \mathbf{U})^{-1} \mathbf{D}^{-1}]_{\bar{i}j} dz = \frac{1}{2d_i d_j \pi i} \int_{p(\Gamma)} (\mathbf{D}^{-1} + \mathbf{U}^* G(z) \mathbf{U})_{\bar{i}j}^{-1} \frac{dz}{z}, \quad (6.7)$$

where \bar{i}, \bar{j} are defined as

$$\bar{i} := r + i, \quad \bar{j} := r + j.$$

Next we decompose $\mathbf{D}^{-1} + \mathbf{U}^* G(z) \mathbf{U}$ by

$$\mathbf{D}^{-1} + \mathbf{U}^* G(z) \mathbf{U} = \mathbf{D}^{-1} + \mathbf{U}^* \Pi(z) \mathbf{U} - \Delta(z), \quad \Delta(z) = \mathbf{U}^* \Pi(z) \mathbf{U} - \mathbf{U}^* G(z) \mathbf{U}. \quad (6.8)$$

It is notable that $\Delta(z)$ can be controlled by the anisotropic law Lemma 4.15. Using the resolvent expansion to the order of one on (6.8), we have

$$\langle v_i, \mathbf{P}_r v_j \rangle = \frac{1}{d_i d_j} (S^{(0)} + S^{(1)} + S^{(2)}), \quad (6.9)$$

where

$$\begin{aligned} S^{(0)} &:= \frac{1}{2\pi i} \int_{p(\Gamma)} \left(\frac{1}{\mathbf{D}^{-1} + \mathbf{U}^* \Pi(z) \mathbf{U}} \right)_{ij} \frac{dz}{z}, \\ S^{(1)} &= \frac{1}{2\pi i} \int_{p(\Gamma)} \left[\frac{1}{\mathbf{D}^{-1} + \mathbf{U}^* \Pi(z) \mathbf{U}} \Delta(z) \frac{1}{\mathbf{D}^{-1} + \mathbf{U}^* \Pi(z) \mathbf{U}} \right]_{ij} \frac{dz}{z}, \\ S^{(2)} &= \frac{1}{2\pi i} \int_{p(\Gamma)} \left[\frac{1}{\mathbf{D}^{-1} + \mathbf{U}^* \Pi(z) \mathbf{U}} \Delta(z) \frac{1}{\mathbf{D}^{-1} + \mathbf{U}^* \Pi(z) \mathbf{U}} \Delta(z) \frac{1}{\mathbf{D}^{-1} + \mathbf{U}^* G(z) \mathbf{U}} \right]_{ij} \frac{dz}{z}. \end{aligned} \quad (6.10)$$

By an elementary computation, we have

$$(\mathbf{D}^{-1} + \mathbf{U}^* \Pi(z) \mathbf{U})_{ij}^{-1} = \begin{cases} \delta_{ij} \frac{z m_{2c}(z)}{z m_{1c}(z) m_{2c}(z) - d_i^{-2}}, & 1 \leq i, j \leq r; \\ \delta_{ij} \frac{z m_{1c}(z)}{z m_{1c}(z) m_{2c}(z) - d_i^{-2}}, & r \leq i, j \leq 2r; \\ \delta_{\bar{i}j} (-1)^{i+j} \frac{z^{1/2} d_i^{-1}}{z m_{1c}(z) m_{2c}(z) - d_i^{-2}}, & 1 \leq i \leq r, r \leq j \leq 2r; \\ \delta_{\bar{i}\bar{j}} (-1)^{i+j} \frac{z^{1/2} d_j^{-1}}{z m_{1c}(z) m_{2c}(z) - d_j^{-2}}, & r \leq i \leq 2r, 1 \leq j \leq r. \end{cases} \quad (6.11)$$

Using the fact $p_i m_{1c}(p_i) m_{2c}(p_i) = \frac{1}{d_i^2}$ and the residual theorem, we have

$$S^{(0)} = \delta_{ij} \frac{m_{2c}(p_i)}{\mathcal{T}'(p_i)} = \delta_{ij} \frac{d_i^4 - c^{-1}}{d_i^2 + 1}. \quad (6.12)$$

Next we will control the term $S^{(1)}$. Applying (6.11), we have

$$\begin{aligned} & \left[\frac{1}{\mathbf{D}^{-1} + \mathbf{U}^* \Pi(z) \mathbf{U}} \Delta(z) \frac{1}{\mathbf{D}^{-1} + \mathbf{U}^* \Pi(z) \mathbf{U}} \right]_{ij} \\ &= \frac{z m_{2c}(z)}{z m_{1c}(z) m_{2c}(z) - d_j^{-2}} \left[\frac{z m_{2c}(z)}{z m_{1c}(z) m_{2c}(z) - d_i^{-2}} \Delta(z)_{ij} + (-1)^{i+\bar{i}} \frac{z^{1/2} d_i^{-1}}{z m_{1c}(z) m_{2c}(z) - d_i^{-2}} \Delta(z)_{\bar{i}j} \right] \\ &+ (-1)^{j+\bar{j}} \frac{z^{1/2} d_j^{-1}}{z m_{1c}(z) m_{2c}(z) - d_j^{-2}} \left[\left(\frac{z m_{2c}(z)}{z m_{1c}(z) m_{2c}(z) - d_i^{-2}} \Delta(z)_{i\bar{j}} + (-1)^{i+\bar{i}} \frac{z^{1/2} d_i^{-1}}{z m_{1c}(z) m_{2c}(z) - d_i^{-2}} \Delta(z)_{\bar{i}\bar{j}} \right) \right]. \end{aligned} \quad (6.13)$$

Therefore, we can write

$$S^{(1)} = \frac{1}{2\pi i} \int_{p(\Gamma)} \frac{f(z)}{(zm_{1c}(z)m_{2c}(z) - d_i^{-2})(zm_{1c}(z)m_{2c}(z) - d_j^{-2})} dz, \quad (6.14)$$

where $f(z) = f_1(z) + f_2(z)$ and $f_{1,2}(z)$ are defined as

$$\begin{aligned} f_1(z) &:= m_{2c}(z)[zm_{2c}(z)\Delta(z)_{ij} + (-1)^{i+\bar{i}}z^{1/2}d_j^{-1}\Delta(z)_{i\bar{j}}], \\ f_2(z) &:= d_i^{-1}[(-1)^{i+\bar{i}}z^{1/2}m_{2c}(z)\Delta(z)_{i\bar{j}} + (-1)^{i+j+\bar{i}+\bar{j}}d_j^{-1}\Delta(z)_{i\bar{j}}]. \end{aligned}$$

We now use the change of variable as in (6.6) and rewrite $S^{(1)}$ as

$$S^{(1)} = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(p(\zeta))}{(\zeta^{-2} - d_i^{-2})(\zeta^{-2} - d_j^{-2})} p'(\zeta) d\zeta = d_i^2 d_j^2 \frac{1}{2\pi i} \int_{\Gamma} \frac{f(p(\zeta))\zeta^4}{(d_i^2 - \zeta^2)(d_j^2 - \zeta^2)},$$

where we use the fact $p(\zeta)m_{1c}(p(\zeta))m_{2c}(p(\zeta)) = \zeta^{-2}$. Recall that

$$p'(\zeta) \sim (\zeta - c^{-1/4}), \quad (6.15)$$

by Lemma 4.8 and Lemma 4.15, we deduce that

$$|f(p(\zeta))| \leq (\zeta - c^{-1/4})^{-1/2} N^{-1/2+\epsilon_1}. \quad (6.16)$$

Denote

$$f_{ij}(\zeta) = \frac{f(p(\zeta))p'(\zeta)\zeta^4}{(d_i + \zeta)(d_j + \zeta)}.$$

f_{ij} is holomorphic inside the contour Γ . By Cauchy's differentiation formula, we have

$$f'_{ij}(\zeta) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{f_{ij}(\xi)}{(\xi - \zeta)^2} d\xi, \quad (6.17)$$

where the contour \mathcal{C} is the circle of radius $\frac{|\zeta - c^{-1/4}|}{2}$ centered at ζ . Hence, by (6.15), (6.16), (6.17) and the residual theorem, we have

$$|f'_{ij}(\zeta)| \leq (\zeta - c^{-1/4})^{-1/2} N^{-1/2+\epsilon_1}. \quad (6.18)$$

In order to estimate $S^{(1)}$, we will consider the following three cases (i) $i, j \in A$, (ii) $i \in A, j \notin A$, (or $i \notin A, j \in A$), (iii) $i, j \notin A$. By the residual theorem, $S^{(1)} = 0$ when case (iii) happens. Hence, we only need to consider the cases (i) and (ii). For the case (i), when $i \neq j$, by the residual theorem and (6.18), we have

$$|S^{(1)}| = d_i^2 d_j^2 \left| \frac{f_{ij}(d_i) - f_{ij}(d_j)}{d_i - d_j} \right| \leq \frac{d_i^2 d_j^2}{|d_i - d_j|} \left| \int_{d_i}^{d_j} |f'_{ij}(t)| dt \right| \leq \frac{d_i^2 d_j^2}{(d_i - c^{-1/4})^{1/2} + (d_j - c^{-1/4})^{1/2}} N^{-1/2+\epsilon_1}.$$

When $i = j$, by the residual theorem, we have

$$|S^{(1)}| \leq d_i^4 (d_i - c^{-1/4})^{-1/2} N^{-1/2+\epsilon_1}.$$

For the case (ii), when $i \in A, j \notin A$, by the residual theorem, we have

$$|S^{(1)}| = \left| \frac{d_i^2 d_j^2 f_{ij}(d_i)}{d_i - d_j} \right| \leq \frac{d_i^2 d_j^2 (d_i - c^{-1/4})^{1/2}}{|d_i - d_j|} N^{-1/2+\epsilon_1}.$$

We can get similar results when $i \notin A, j \in A$. Putting all the cases together, we find that there exists some constant $C > 0$, such that

$$\begin{aligned} |S^{(1)}| \leq C N^{-1/2+\epsilon_1} & \left[\frac{\mathbf{1}(i \in A, j \in A) d_i^2 d_j^2}{(d_i - c^{-1/4})^{1/2} + (d_j - c^{-1/4})^{1/2}} + \mathbf{1}(i \in A, j \notin A) \frac{d_i^2 d_j^2 (d_i - c^{-1/4})^{1/2}}{|d_i - d_j|} \right. \\ & \left. + \mathbf{1}(i \notin A, j \in A) \frac{d_i^2 d_j^2 (d_i - c^{-1/4})^{1/2}}{|d_i - d_j|} \right]. \quad (6.19) \end{aligned}$$

Finally, we need to estimate $S^{(2)}$. Here the residual calculations can not be applied directly as $\mathbf{U}^*G(z)\mathbf{U}$ is not necessary to be diagonal and a relation comparable to $p(\zeta)m_{1c}(p(\zeta))m_{2c}(p(\zeta)) = \zeta^{-2}$ does not exist. Instead, we need to precisely choose the contour Γ . A crucial estimate is the following lemma, which can be found in [5, Lemma 5.6]. Define the boundary of $B_{\rho_k}(d_k)$ as $\partial B_{\rho_k}(d_k)$, then we have

Lemma 6.2. *Denote*

$$\Gamma_k = \Gamma \cap \partial B_{\rho_k}(d_k),$$

then for $k \in A$, and $\zeta \in \Gamma_k$, recall (6.2), we have

$$|\zeta - d_l| \sim \rho_k + |d_k - d_l|, \quad 1 \leq l \leq r. \quad (6.20)$$

By (6.1), the fact r is finite and (6.15), it is easy to check that

$$|S^{(2)}| \leq \int_{\Gamma} \frac{d_i^2 d_j^2 N^{-1+2\epsilon_1}}{|\zeta - d_i||\zeta - d_j|} \left\| \frac{1}{\mathbf{D}^{-1} + \mathbf{U}^*G(p(\zeta))\mathbf{U}} \right\| |d\zeta|. \quad (6.21)$$

We now assume for $\zeta \in \Gamma_k$, by the resolvent expansion, we have

$$\begin{aligned} (\mathbf{D}^{-1} + \mathbf{U}^*G(p(\zeta))\mathbf{U})^{-1} &= (\mathbf{D}^{-1} + \mathbf{U}^*\Pi(p(\zeta))\mathbf{U})^{-1} \\ &\quad + (\mathbf{D}^{-1} + \mathbf{U}^*\Pi(p(\zeta))\mathbf{U})^{-1}(\mathbf{U}^*G(p(\zeta))\mathbf{U} - \mathbf{U}^*\Pi(p(\zeta))\mathbf{U})(\mathbf{D}^{-1} + \mathbf{U}^*G(p(\zeta))\mathbf{U})^{-1}. \end{aligned} \quad (6.22)$$

By (6.1), we have

$$\|\mathbf{U}^*G(p(\zeta))\mathbf{U} - \mathbf{U}^*\Pi(p(\zeta))\mathbf{U}\| \leq |p(\zeta) - \lambda_+|^{-1/4} N^{-1/2+\epsilon_1} \leq (d_k - c^{-1/4})^{-1/2} N^{-1/2+\epsilon_1}. \quad (6.23)$$

For $1 \leq l \leq r$, there exists some constant $c > 0$, such that

$$|T(p(\zeta)) - d_l^{-2}| \geq c(|\zeta - d_l| \wedge c^{-1/4}) \geq c|\zeta - d_k| = c\rho_k \geq c(d_k - 1)^{-1/2} N^{-1/2+\epsilon_0}, \quad (6.24)$$

where ϵ_0 is defined in (3.4) and in the last step we use (6.3). Hence, by (6.11) and the fact r is finite, we have

$$\|(\mathbf{D}^{-1} + \mathbf{U}^*\Pi(p(\zeta))\mathbf{U})^{-1}\| \leq \frac{C}{\rho_k}, \quad (6.25)$$

for some constant $C > 0$. Therefore, by (6.22), (6.23) and (6.25), we have

$$\|(\mathbf{D}^{-1} + \mathbf{U}^*G(p(\zeta))\mathbf{U})^{-1}\| \leq \frac{C}{\rho_k}. \quad (6.26)$$

Decomposing Γ into $\Gamma = \cup_{k \in A} \Gamma_k$, by (6.20), (6.21), (6.26) and Γ_k has length $2\pi\rho_k$, we have

$$|S^{(2)}| \leq C \sum_{k \in A} \sup_{\zeta \in \Gamma_k} \frac{d_i^2 d_j^2 N^{-1+2\epsilon_1}}{|\zeta - d_i||\zeta - d_j|} \leq C \sum_{k \in A} \frac{d_i^2 d_j^2 N^{-1+2\epsilon_1}}{(\rho_k + |d_k - d_i|)(\rho_k + |d_k - d_j|)}. \quad (6.27)$$

for some constant C . To estimate the right-hand side of (6.27), for $i \notin A$, by (6.2), we have that

$$\rho_k \leq d_k - c^{-1/4} \leq |d_k - c^{-1/4} + c^{-1/4} - d_i| \leq |d_k - d_i|,$$

from which we conclude

$$\sum_{k \in A} \frac{1}{(\rho_k + |d_k - d_i|)^2} \leq \sum_{k \in A} \frac{1}{|d_k - d_i|^2} \leq \frac{C}{\nu_i(A)},$$

where we use the fact that r is finite. Similarly, for $i \in A$, by (6.2), we have $|d_k - d_i| \leq \rho_k$. Combining with the fact $\rho_k + |d_i - d_k| \geq \rho_i$ for all $k \in A$, we have

$$\sum_{k \in A} \frac{1}{(\rho_k + |d_k - d_i|)^2} \leq \frac{C}{\rho_i^2} \leq \frac{C}{\nu_i^2(A)} + \frac{C}{(d_i - c^{-1/4})^2},$$

for some constant $C > 0$. Combine with (6.27), we have

$$|S^{(2)}| \leq C d_i^2 d_j^2 N^{-1+2\epsilon_1} \left(\frac{1}{\nu_i} + \frac{\mathbf{1}(i \in A)}{|d_i - c^{-1/4}|} \right) \left(\frac{1}{\nu_j} + \frac{\mathbf{1}(j \in A)}{|d_j - c^{-1/4}|} \right), \quad (6.28)$$

for some constant $C > 0$. Therefore, plugging (6.12), (6.19) and (6.28) into (6.9), we conclude the proof of (3.9). Before concluding this section, we briefly discuss the proof of (3.8). Instead of considering (6.5), we will analyse

$$u_i^* \tilde{\mathcal{G}}_1 u_j = \mathbf{u}_i^* \tilde{\mathcal{G}} \mathbf{u}_j,$$

where $\mathbf{u}_i, \mathbf{u}_j$ are the natural emdedding of u_i, u_j . Then by Lemma 4.12 and Cauchy's integral formula , we have

$$\langle u_i, \mathbf{P}_l u_j \rangle = \frac{1}{2\pi i} \int_{p(\Gamma)} [D^{-1}(D^{-1} + \mathbf{U}^* G(z) \mathbf{U})^{-1} D^{-1}]_{ij} dz = \frac{1}{2d_i d_j \pi i} \int_{p(\Gamma)} (D^{-1} + \mathbf{U}^* G(z) \mathbf{U})_{\bar{i}j}^{-1} \frac{dz}{z}.$$

Then following a similar discussion, to be specific, decompose the above integration into three parts (see (6.9)), compute the convergent limit from $S^{(1)}$ and control the bounds for $S^{(2)}, S^{(3)}$, we can finish the proof. We remark that the convergent limit is different because we will use $(\mathbf{D}^{-1} + \mathbf{U}^* \Pi(z) \mathbf{U})_{ij}, r \leq i, j \leq 2r$ in (6.11), which leads to

$$S^{(0)} = \delta_{ij} \frac{m_{1c}(p_i)}{\mathcal{T}'(p_i)}.$$

This concludes the proof of (3.8). \square

6.2 Non-outlier singular vectors

For the non-outliers, the proof strategy in Section 6.1 will not work as we cannot use the residual theorem. This is due to the fact that there are no poles inside Γ defined in (6.4). Instead, we will use a spectral decomposition for our proofs. Due to similarity, we only prove (3.11) and point out its differences from (3.10).

Proof of (3.11). Denote

$$z = \mu_j + i\eta, \quad (6.29)$$

where η is defined as the smallest solution of

$$\text{Im } m_{2c}(z) = N^{-1+6\epsilon_1} \eta^{-1}.$$

Therefore, by Lemma 4.13 and Lemma 4.15, we have

$$| \langle \mathbf{u}, \underline{\Sigma}^{-1}(G(z) - \Pi(z)) \underline{\Sigma}^{-1} \mathbf{v} \rangle | \leq \frac{N^{3\epsilon_1}}{N\eta}. \quad (6.30)$$

And by Lemma 4.8 and (3.7), we deduce that (see [5, (6.5) and (6.6)])

$$\eta \sim \begin{cases} \frac{N^{6\epsilon_1}}{N\sqrt{\kappa} + N^{2/3+3\epsilon_1}}, & \text{if } \mu_j \leq \lambda_+ + N^{-2/3+4\epsilon_1}, \\ N^{-1/2+3\epsilon_1} \kappa^{1/4}, & \text{if } \mu_j \geq \lambda_+ + N^{-2/3+4\epsilon_1}. \end{cases} \quad (6.31)$$

For z defined in (6.29), by the spectral decomposition, we have

$$\langle v_i, \tilde{v}_j \rangle^2 \leq \eta \langle v_i, \text{Im } \tilde{\mathcal{G}}_2(z) v_i \rangle = \eta \langle \mathbf{v}_i, \text{Im } \tilde{\mathcal{G}}(z) \mathbf{v}_i \rangle,$$

where $\mathbf{v}_i \in \mathbb{R}^{M+N}$ is the natural embedding of v_i . By Lemma 4.12, we have

$$\langle \mathbf{v}_i, \tilde{\mathcal{G}}(z) \mathbf{v}_i \rangle = [\mathbf{D}^{-1} - \mathbf{D}^{-1}(\mathbf{D}^{-1} + \mathbf{U}^* G(z) \mathbf{U})^{-1} \mathbf{D}^{-1}]_{\bar{i}i} = -[\mathbf{D}^{-1}(\mathbf{D}^{-1} + \mathbf{U}^* G(z) \mathbf{U})^{-1} \mathbf{D}^{-1}]_{\bar{i}i}.$$

Similar to (6.7), it is easy to check

$$\langle \mathbf{v}_i, \tilde{\mathcal{G}}(z) \mathbf{v}_i \rangle = -\frac{1}{z d_i^2} (\mathbf{D}^{-1} + \mathbf{U}^* G(z) \mathbf{U})_{ii}^{-1}.$$

Similar to (6.9), using a simple resolvent expansion, we have

$$\begin{aligned} & \langle \mathbf{v}_i, \tilde{G}(z) \mathbf{v}_i \rangle \\ &= -\frac{1}{zd_i^2} \left[\frac{zm_{2c}(z)}{zm_{1c}(z)m_{2c}(z) - d_i^{-2}} + \frac{zf(z)}{(zm_{1c}(z)m_{2c}(z) - d_i^{-2})^2} + ([(\mathbf{D}^{-1} + \mathbf{U}^* \Pi(z) \mathbf{U})^{-1} \Delta(z)]^2 (\mathbf{D}^{-1} + \mathbf{U}^* G(z) \mathbf{U})^{-1})_{ii} \right], \end{aligned} \quad (6.32)$$

where $f(z)$ is defined in (6.14) and we use (6.11) and (6.13). To estimate the right-hand side of (6.32), we use the following error estimate

$$\min_j |d_j^{-2} - \mathcal{T}(z)| \geq \text{Im } \mathcal{T}(z) \sim \text{Im } m_{2c}(z) = \frac{N^{6\epsilon_1}}{N\eta} \gg \frac{N^{3\epsilon_1}}{N\eta} \geq |\Delta(z)|,$$

where we use (6.30). By a similar resolvent expansion as (6.22), there exists some constant $C > 0$, such that

$$\left\| \frac{1}{\mathbf{D}^{-1} + \mathbf{U}^* G(z) \mathbf{U}} \right\| \leq \frac{C}{\text{Im } m_{2c}(z)} = CN^{1-6\epsilon_1} \eta.$$

We therefore get from (6.32), the definition of f and (6.30) that

$$\langle \mathbf{v}_i, \tilde{G}(z) \mathbf{v}_i \rangle = \frac{m_{2c}(z)}{1 - d_i^2 \mathcal{T}(z)} + O\left(\frac{d_i^2}{|1 - d_i^2 \mathcal{T}(z)|^2} \frac{N^{2\epsilon_1}}{N\eta}\right). \quad (6.33)$$

By (6.33), we have

$$\langle v_i, \tilde{v}_j \rangle^2 \leq \frac{\eta}{|1 - d_i^2 \mathcal{T}(z)|^2} \left[\text{Im } m_{2c}(z) (1 - d_i^2 c^{1/2} + \text{Re}(d_i^2 c^{1/2} - d_i^2 \mathcal{T}(z)) + \text{Re } m_{2c}(z) \text{Im}(1 - d_i^2 \overline{\mathcal{T}(z)}) + \frac{Cd_i^2 N^{2\epsilon_1}}{N\eta} \right].$$

By (6.31), we have

$$\text{Im } m_{2c}(z) [(1 - d_i^2 c^{1/2}) + \text{Re}(d_i^2 c^{1/2} - d_i^2 \mathcal{T}(z))] \leq CN^{-1+6\epsilon_1} \eta^{-1} (1 + \sqrt{\kappa + \eta}),$$

where we use (4.23). As a consequence, we have

$$\frac{\eta}{|1 - d_i^2 \mathcal{T}(z)|^2} \text{Im } m_{2c}(z) [(1 - d_i^2 c^{1/2}) + \text{Re}(d_i^2 c^{1/2} - d_i^2 \mathcal{T}(z))] \leq C \frac{N^{6\epsilon_1}}{N|1 - d_i^2 \mathcal{T}(z)|^2}.$$

For the other item, by Lemma 4.8, Lemma 4.9 and (6.29), we have

$$|\text{Re } m_{2c}(z) \text{Im } \mathcal{T}(z)| \sim \text{Im } m_{2c}(z).$$

Therefore, we have

$$\left| \frac{\eta}{|1 - d_i^2 \mathcal{T}(z)|^2} \text{Re } m_{2c}(z) \text{Im}(1 - d_i^2 \overline{\mathcal{T}(z)}) \right| \leq C \frac{N^{6\epsilon_1}}{N|1 - d_i^2 \mathcal{T}(z)|}.$$

Putting all these estimates together, we have

$$\langle v_i, \tilde{v}_j \rangle^2 \leq \frac{CN^{6\epsilon_1} + Cd_i^2 N^{2\epsilon_1}}{N|1 - d_i^2 \mathcal{T}(z)|^2}.$$

Next we will give an estimator of the denominator $1 - d_i^2 \mathcal{T}(z)$. We first observe that

$$|1 - d_i^2 \mathcal{T}(z)|^2 = d_i^2 |d_i^{-2} - c^{1/2} + c^{1/2} - \mathcal{T}(z)|^2.$$

In the case $|d_i - c^{-1/4}| \leq \frac{1}{2}$, we have that (see the equation above (6.11) of [16])

$$|1 - d_i^2 \mathcal{T}(z)|^2 \geq d_i^2 [(d_i^{-2} - c^{1/2} - |\text{Re } \mathcal{T}(z) - c^{1/2}|)_+ + \text{Im } \mathcal{T}(z)].$$

By [5, (6.11)], we have that for any $y \leq tx$, $t \geq 1$,

$$(x - y)_+ + z \geq \frac{x}{3t} + \frac{z}{3}. \quad (6.34)$$

For $\mu_i \in [\lambda_+ - N^{-2/3+C\epsilon_0}, \lambda_+]$, using $t = C$ in (6.34) and (6.31), we find that there exists some constant $c > 0$, such that

$$|1 - d_i^2 \mathcal{T}(z)|^2 \geq cd_i^2 \left(|d_i^{-2} - c^{1/2}| + \text{Im } \mathcal{T}(z) \right).$$

When $\mu_j \in [\lambda_+, \lambda_+ + N^{-2/3+C\epsilon_0}]$, choosing $t = K^{2\epsilon_0}$ in (6.34) and using (6.31), we get

$$|1 - d_i^2 \mathcal{T}(z)|^2 \geq cd_i^2 \left(N^{-2\epsilon_0} |d_i^{-2} - c^{1/2}| + \text{Im } \mathcal{T}(z) \right).$$

For the case $|d_i - c^{-1/4}| \geq \frac{1}{2}$, by Lemma 4.8 and Lemma 4.9, we have

$$|1 - d_i^2 \mathcal{T}(z)|^2 \geq cd_i^2 (|d_i^{-2} - c^{1/2}| + \text{Im } \mathcal{T}(z)).$$

Therefore, we have

$$\langle v_i, \tilde{v}_j \rangle^2 \leq \frac{N^{C\epsilon_0}}{N((d_i - c^{-1/4})^2 + \kappa_j)}, \quad \kappa_j := |\mu_j - \lambda_+|,$$

where we use the fact that $\text{Im } \mathcal{T}(z) \geq c\sqrt{\kappa_j}$ (see [5, (6.14) and (6.15)]). This concludes the proof of (3.11). For the proof of (3.10), we will use the spectral decomposition

$$\langle u_i, \tilde{u}_j \rangle^2 \leq \eta \langle u_i, \text{Im } \tilde{\mathcal{G}}_1(z) u_i \rangle = \eta \langle \mathbf{u}_i, \text{Im } \tilde{G}(z) \mathbf{u}_i \rangle,$$

and

$$\langle \mathbf{u}_i, \tilde{G}(z) \mathbf{u}_i \rangle = -\frac{1}{zd_i^2} (\mathbf{D}^{-1} + \mathbf{U}^* G(z) \mathbf{U})_{ii}^{-1}.$$

Then by the resolvent expansion similar to (6.32) and control the items using Lemma 4.8, Lemma 4.9, Lemma 4.13 and Lemma 4.15, we can conclude our proof. \square

Acknowledgments. The author would like to thank Jeremy Quastel, Bálint Virág and Zhou Zhou for fruitful discussions and valuable suggestions, which have significantly improved the paper. The results have been presented at the Toronto probability seminar. For the purpose of reproducible research, we offer R codes on our website <http://individual.utoronto.ca/xcd/> to recover our algorithm and simulations in Section 2.

References

- [1] O. Alter, P. Brown, and D. Botstein. Singular value decomposition for genome-wide expression data processing and modeling. *Proc. Natl. Acad. Sci. USA*, 97:10101–10106, 2000.
- [2] J. Baik, G. Ben Arous, and S. Péché. Phase transition of the largest eigenvalue for nonnull complex sample covariance matrices. *Ann. Prob.*, 33:1643–1697, 2005.
- [3] F. Benaych-Georges and R. Nadakuditi. The singular values and vectors of low rank perturbations of large rectangular random matrices. *J. Multivar. Anal.*, 227:494–521, 2011.
- [4] A. Bloemendal, L. Erdős, A. Knowles, H. T. Yau, and J. Yin. Isotropic local laws for sample covariance and generalized Wigner matrices. *Electron. J. Probab.*, 19:1–53, 2014.
- [5] A. Bloemendal, A. Knowles, H. T. Yau, and J. Yin. On the principal components of sample covariance matrices. *Prob. Theor. Rel. Fields*, 164:459–552, 2016.
- [6] J. Bun, R. Allez, J. Bouhauud, and M. Potters. Rotational invariant estimator for general noisy matrices. *IEEE Trans. Inf. Theory*, 62:7475–7490, 2016.

- [7] X. Ding. Singular vector distribution of covariance matrices. *arXiv: 1611.01837*.
- [8] X. Ding and F. Yang. A necessary and sufficient condition for edge universality at the largest singular values of covariance matrices. *arXiv: 1607.06873*.
- [9] D. Donoho. De-noising by soft-thresholding. *IEEE Trans. Inf. Theory*, 41:613–627, 1995.
- [10] M. Elad. *Sparse and redundant representations: from theory to applications in signal and image processing*. Springer, 2010.
- [11] M. Gavish and D. Donoho. The optimal hard threshold for singular values is $4/\sqrt{3}$. *IEEE Trans. Inf. Theory*, 60:5040–5053, 2014.
- [12] M. Gavish and D. Donoho. Optimal shrinkage of singular values. *IEEE Trans. Inf. Theory*, 63:2137–2152, 2017.
- [13] G. Golub and C. van Loan. *Matrix computation, 3rd edition*. The Johns Hopkins University Press, 1996.
- [14] G. James, D. Witten, T. Hastie, and R. Tibshirani. *An Introduction to Statistical Learning*. Springer, 2013.
- [15] A. Knowles and J. Yin. Anisotropic local laws for random matrices. *arXiv:1410.3516*.
- [16] A. Knowles and J. Yin. The isotropic semicircle law and deformation of Wigner matrices. *Comm. Pure Appl. Math.*, 11:1663–1749, 2013.
- [17] L. Laloux, P. Cizeau, M. Potters, and J. Bouchaud. Random matrix theory and financial correlations. *Int. J. Theor. Appl. Finan.*, 3:391–397, 2000.
- [18] M. Lee, H. Shen, J. Huang, and J. Marron. Biclustering via sparse singular value decomposition. *Biometrics*, 66:1087–1095, 2010.
- [19] V. A. Marčenko and L. A. Pastur. Distribution of eigenvalues for some sets of random matrices. *Mathematics of the USSR-Sbornik*, 1:457, 1967.
- [20] D. Paul. Asymptotics of sample eigenstructure for a large dimensional spiked covariance model. *Statistica Sinica*, 17:1617–1642, 2007.
- [21] B. Pontes, R. Giráldez, and J. Aguilar-Ruiz. Biclustering on expression data: A review. *J. Biomed. Inform.*, 57:163–180, 2015.
- [22] J. W. Silverstein. The Stieltjes transform and its role in eigenvalue behavior of large dimensional random matrices. *Random Matrix Theory and its Applications, Lecture Notes Series*. World Scientific, Singapore, 2009.
- [23] T. Tao. *Topics in Random Matrix Theory*. American Mathematical Society, 2012.
- [24] C. Tracy and H. Widom. On orthogonal and symplectic matrix ensembles. *Comm. Math. Phys.*, 177:727–754, 1996.
- [25] D. Tufts and A. Shah. Estimation of a signal waveform from noisy data using low-rank approximation to a data matrix. *IEEE Trans. Sig. Proc.*, 41:7475–7490, 1993.
- [26] D. Yang, Z. Ma, and A. Buja. Rate optimal denoising of simultaneously sparse and low rank matrices. *J. Mach. Learn. Res.*, 17:1–27, 2016.